Ma1023 | **Calculus III** | A Term, 2013

Final Exam (A) | Print Name: ____________________

1. (6 pts) For any 10 of the following, mark with **T** and **F** for false, and **X** if it cannot be determined from the given information.

a. ____ For all vectors $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

True. The cross product is anti-commutative.

b. ____ If $\lim_{n\to\infty} \dfrac{f(n+1)}{f(n)} = 0$ then the series $\sum_{n=0}^{\infty} f(n)$ converges.

True. Ratio Test.

c. ____ If the power series $\sum_{n=0}^{\infty} c_k(x-8)^k$ converges absolutely at $x = 7$, then it converges absolutely for $x = 5$.

You cannot conclude it. The radius of convergence could be 2.

d. ____ For all vectors $\mathbf{u}$, it is true that $\mathbf{u} \times \mathbf{0} = \mathbf{u}$.

False. It is only true for $\mathbf{u} = \mathbf{0}$.

e. ____ The Taylor series for $f(x) = \cos(x)$ has radius of convergence 1.

False. It is infinite.

f. ____ The plane $(2, 1, 2) \cdot \mathbf{v} = 3$ has distance 1 from the origin.

True. $(2/3, 1/3, 2/3)$ is a unit vector.

2. (8 pts) For each value of x determine whether the following power series converges absolutely, converges conditionally, or diverges. Show all steps required to carefully present your work.

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-1)^n$$

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n\,2^n} \qquad \text{Absolute Value Series } \sum_{n=1}^{\infty} \frac{|x-1|^n}{n\,2^n}$$

Ratio Test: $\displaystyle \lim_{n\to\infty} \frac{\left(\frac{|x-1|^{n+1}}{(n+1)\,2^{n+1}}\right)}{\left(\frac{|x-1|^n}{n\,2^n}\right)} = \lim_{n\to\infty} \frac{|x-1|\,n}{2\,(n+1)} = \lim_{n\to\infty} \frac{|x-1|}{2}\left(\frac{1}{1-\frac{1}{n}}\right) = \frac{|x-1|}{2}.$

Series Converges if $\frac{|x-1|}{2} < 1$ or $|x-1| < 2$

$-2 < x-1 < 2$

Converges absolutely for $-1 < x < 3$

Diverges for $x > 3$ and $x < -1$

Endpoints: $x=3$, $\displaystyle \sum_{n=1}^{\infty} \frac{(3-1)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic Series) p-series $p=1$ diverges

$x=-1$ $\displaystyle \sum_{n=1}^{\infty} \frac{(-1-1)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Does not converge absolutely ($x=3$)

Alternating Series Test:

* $\frac{1}{n} \geq 0$ It alternates
* $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2} < 0$, so $\frac{1}{n}$ decreasing
* $\lim_{n\to\infty} \frac{1}{n} = 0$

So the series converges by the alternating series test.

So at $x=-1$ series converges conditionally.

Summary
$-1 < x < 3$ Series converges absolutely
$x = -1$ Series converges conditionally
$x \geq 3$, $x < -1$ Series Diverges

3. (4 pts) The power series $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ converges absolutely for all x, so define

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$$

Answer each of the following:
a) Find $f^{(7)}(0)$.

b) Find the Taylor Series for $f'(x)$ centered at 0. What is its radius of convergence?

a) Taylor Series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$

7th term centered at 0 $\frac{f^{(7)}(0)\,x^7}{7!}$ For $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ the term in x^7 has $2n+1=7$, $n=3$

$$\frac{f^{(7)}(0)\,x^7}{7!} = \frac{x^7}{3!}$$

$$f^{(7)}(0) = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$$

b) Differentiate term by term $\frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^{2n+1}}{n!}\right)$

$$= \sum_{n=0}^{\infty} \frac{(2n+1)\,x^{2n}}{n!}$$ (note here was no constant term to discard)

4. (4 pts) Using the known Taylor Series for e^x with center 0, find the Taylor series for the function $f(x) = x^2 e^{3x+1}$ centered at 0.

Show all steps required to carefully present your work.

Recall $e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!}$

$$x^2 e^{3x+1} = x^2 e^{3x} e = x^2 \sum_{k=0}^{\infty} \frac{(3x)^n}{n!} e = \sum_{k=0}^{\infty} \frac{e\,3^n x^n x^2}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{e\,3^n x^{n+2}}{n!}$$

5. (8 pts) Let $\mathbf{f}(t) = \cos(\pi t^2)\mathbf{i} + \sin(\pi t^2)\mathbf{j} + t^2\mathbf{k}$ be a position vector at time t.

a) Find all times where the speed is 0.

b) Show that the angle between the velocity vector $\mathbf{v}(t)$ and the vector $\mathbf{k}$ is constant for all $t > 0$.

c) Find a unit vector tangent to the curve at $t = 2$.

d) Compute $|\mathbf{f}(2) \times \mathbf{v}(2)|$.

a) $\vec{f}(t) = \cos(\pi t^2)\vec{i} + \sin(\pi t^2)\vec{j} + t^2\vec{k}$

Velocity:

$v(t) = -2\pi t \sin(\pi t^2)\vec{i} + 2\pi t \cos(\pi t^2)\vec{j} + 2t\vec{k}$

Speed

$\frac{ds}{dt} = \sqrt{4\pi^2 t^2 \sin^2(\pi t^2) + 4\pi^2 t^2 \cos^2(\pi t^2) + 4t^2}$

$= 2|t|\sqrt{\pi^2(\sin^2(\pi t^2) + \cos^2(\pi t^2)) + 1}$

$= 2|t|\sqrt{\pi^2+1}$

So the speed is zero only at $t=0$.

b) Angle between $\vec{v}(t)$ and $\vec{k} = (0,0,1)$

$\vec{v}\cdot\vec{k} = |\vec{v}||\vec{k}|\cos(\theta)$

$2t = (2|t|\sqrt{\pi^2+1})\,1\cos(\theta)$, for $t>0$ $|t|=t$

$\cos(\theta) = \frac{1}{\sqrt{\pi^2+1}}$ which does not depend on t.

c) $\vec{v}(2) = -4\pi \sin(4\pi)\vec{i} + 4\pi\cos(4\pi)\vec{j} + 4\vec{k}$

$= 4\pi\vec{j} + 4\vec{k}$

$\frac{\vec{v}(2)}{|\vec{v}(2)|} = \frac{4\pi\vec{j} + 4\vec{k}}{4\sqrt{\pi^2+1}} = \frac{\pi}{\sqrt{\pi^2+1}}\vec{j} + \frac{1}{\sqrt{\pi^2+1}}\vec{k}$

d) $\vec{f}(2) \times \vec{v}(2) = (\cos(4\pi)\vec{i} + \sin(4\pi)\vec{j} + 4\vec{k}) \times (4\pi\vec{j} + 4\vec{k})$

$= (\vec{i} + 4\vec{k}) \times (4\pi\vec{j} + 4\vec{k})$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 4 \\ 0 & 4\pi & 4 \end{vmatrix} = -16\pi\vec{i} - 4\vec{j} + 4\pi\vec{k}$$

$|\vec{f}(2) \times \vec{v}(2)| = \sqrt{16^2\pi^2 + 4^2 + 4^2\pi^2} = 4\sqrt{5\pi^2+1}$

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