



MA1021

Calculus I

B Term, 2014

Final Exam

Print Name: _____

Sign: _____

Write up your solutions neatly and show all your work.

1. (28 pts) Compute each of the following derivatives: You do not have to simplify your results, but they must be neat, clear and correctly written up.

a) $\frac{d}{dx} (5x^3 - 3x^5 + \pi x^{\pi-1} + \pi e^{\pi-1}) =$

$5 \cdot 3x^2 - 3 \cdot 5x^4 + \pi(\pi - 1)x^{\pi-2} + 0$ (just the power rule.)

b) $\frac{d}{dx} [(1 + x^2 + x^4 + x^6)(x + x^3 + x^5 + x^7)] =$

$(1 + x^2 + x^4 + x^6)(1 + 3x^2 + 5x^4 + 7x^6) + (x + x^3 + x^5 + x^7)(0 + 2x + 4x^3 + 6x^5)$
(multiplying out and using the power rule is much slower. you can use also the product rule on $x(1 + x^2 + x^4 + x^6)^2$.)

c) $\frac{d}{dx} [\sin(x) + \cos(2x) - \tan(3x)] =$

$\frac{d}{dx} [\cos(x) - 2\sin(2x) - 3\sec^2(3x)]$ trig rules and chain rule.

d) $\frac{d}{dx} (x + \sin(x^3))^{15} =$

$15(x + \sin(x^3))^{14} (1 + \frac{d}{dx} \sin(x^3)) =$

$15(x + \sin(x^3))^{14} (1 + \cos(x^3) \frac{d}{dx} (x^3)) =$

$15(x + \sin(x^3))^{14} (1 + (3x^2) \cos(x^3))$

$$\text{e) } \frac{d}{dx} (e^x - 2e^{2x} + e \cdot 2^{ex}) =$$

$$\begin{aligned} & e^x - 4e^{2x} + e \cdot 2^{ex} \ln(2) \frac{d}{dx}(ex) = \\ & e^x - 4e^{2x} + e^2 \cdot 2^{ex} \ln(2) \end{aligned}$$

$$\text{f) } \frac{d}{dx} [\ln(2x) + x \ln(x) + \ln(\sec(x))] =$$

$$\frac{1}{2x} 2 + (x \frac{1}{x} + 1 \cdot \ln(x)) + \frac{1}{\sec(x)} (\sec(x) \tan(x))$$

$$\text{g) } \frac{d}{dx} \sqrt{x^2 + x + 1} =$$

$$\frac{1}{2\sqrt{x^2 + x + 1}} \frac{d}{dx} (x^2 + x + 1) = \frac{2x + 1}{2\sqrt{x^2 + x + 1}}$$

2. (12 points) Let $y = (x^2 - 3)(x^2 - 5)$.

The derivatives are given by $y' = 4x(x - 2)(x + 2)$ and $y'' = 12(x - \sqrt{4/3})(x + \sqrt{4/3})$

a) Find all critical values and determine all intervals where y is increasing.

Inspection with no algebra shows the critical values are -2 , 0 , 2 , and the increasing intervals are $(-2, 0)$ and $(2, \infty)$.

a) Find all inflection values and determine all intervals where y is concave up.

The inflection values are $\pm\sqrt{4/3}$. The intervals where y is concave up are $(-\infty, -\sqrt{4/3})$ and $(\sqrt{4/3}, \infty)$.

c) Make a neat sketch of y on the axis below, labeling all important features.

The critical points are $(-2, -1)$, $(0, 15)$, and $(2, -1)$.

The inflection points are $(-\sqrt{4/3}, 55/9)$ and $(\sqrt{4/3}, 55/9)$.

There are no asymptotes of any kind.

3. (5 points) Use the definition of the derivative to compute the derivative of $f(x) = 1/x$ at $x = 2$.

[You must use the definition. We know from the derivative rules that the answer will be $-1/2^2 = -1/4$.

$$\begin{aligned} f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = \\ \lim_{x \rightarrow 2} \frac{\left(\frac{2-x}{2x}\right)}{x - 2} &= \lim_{x \rightarrow 2} \left(\frac{2-x}{2x(x-2)}\right) = \\ \lim_{x \rightarrow 2} \left(\frac{-1}{2x}\right) &= \left(\frac{-1}{4}\right) = \end{aligned}$$

4. (15 pts) The function $f(x) = \ln(x^8) - x$ is defined on the closed interval $[6, 9]$. Find the maximum and minimum value of $f(x)$ on that interval.

$$f'(x) = \frac{1}{x^8}(8x^7) - 1 = \frac{8}{x} - 1. \text{ So the critical values } f'(x) = 0 \text{ occur when } x = 8.$$

$f'(x)$ is otherwise defined on the interval so there are no other critical values.

$f''(x) = -8/x^2$, which is negative so $f(x)$ is concave down on the interval, so the critical point is the maximum value.

The minimum value occurs at one of the endpoints, either $8 \ln(9) - 9$ or $8 \ln(6) - 6$, and you would need a calculator to determine which.

5. (25 pts) Find each of the following limits:

a) $\lim_{x \rightarrow 3} x^2 - 9$

A polynomial is continuous everywhere, so $3^2 - 9 = 0$.

b) $\lim_{x \rightarrow 3} \frac{x^2 + 4x + 3}{x^2 - 9}$

The limit of the numerator is 48 and the limit of the denominator is 0, so the limit does not converge.

Since $x^2 + 9 > 0$ for $x > 3$ and $x^2 + 9 < 0$ for $x < 3$,

$$\lim_{x \rightarrow 3^-} \frac{x^2 + 4x + 3}{x^2 - 9} = -\infty \text{ and}$$

$$\lim_{x \rightarrow 3^+} \frac{x^2 + 4x + 3}{x^2 - 9} = +\infty$$

So the total limit is neither one.

The limit does not exist.

c) $\lim_{x \rightarrow 0} \frac{x \ln(x + e^2)}{\sin(3x) \cos(x)}$

$$\lim_{x \rightarrow 0} \frac{3x}{\sin(3x)} \cdot \frac{\ln(x + e^2)}{3 \cos(x)} = \lim_{x \rightarrow 0} 1 \cdot \frac{\ln(e^2)}{3 \cdot 1} = 2/3$$

d) $\lim_{x \rightarrow 0^+} \frac{x^2 - 7}{x}$

The numerator approaches -7 , the denominator approaches 0 through positive values since $x > 0$, so the limit is $+\infty$.

e) $\lim_{x \rightarrow \infty} \frac{x^7 - 7x + 4}{1 + 3x^3 + 5x^5 + 7x^7}$

$$\lim_{x \rightarrow \infty} \frac{1 - 7x^{-6} + 4x^{-7}}{x^{-7} + 3x^{-4} + 5x^{-2} + 7} = 1/7$$

6. (7 pts.) Let $f(x) = 9 - x^3$. Find the equation for the tangent line to $f(x)$ at $x = 2$.

$f'(x) = -3x^2$, so the slope of the tangent line at $x = 2$ is $f'(2) = -12$.

The value of f at 2 is $9 - 2^3 = 1$.

The equation of the tangent line is $y = -12(x - 2) + 1$.

7. (8 pts.) Graph the function $f(x)$ below.

Clearly label any point of discontinuity, and state what type of discontinuity it is.

Clearly label any point of non-differentiability.

$$f(x) = \begin{cases} |x| & \text{for } -1 \leq x \\ -1 & \text{for } -3 \leq x < -1 \\ -x & \text{for } x < -3 \end{cases}$$

The graph is the graph of the absolute value function with the section from -3 to -1 replaced by a horizontal line of height -1 .

There are two jump discontinuities at -1 and -3 .

The function $f(x)$ is continuous at the corner of the absolute value at $x = 0$, however the derivative is not defined there, so there is a point of non-differentiability there.