Exercises: Pascal's Triangle

Some of these exercises use the "Hockey Stick Identities"

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}, \qquad \sum_{i=k}^{n} \binom{i}{i-k} = \binom{n+1}{n-k}$$

1. Show by induction that the binomial coefficients are "bimodal", that is, that as a function of k, the value of $\binom{n}{k}$ increases to the middle, and then decreases.

2. Show directly, not by induction, that if that for $k \leq n$ that $\binom{2n}{k-1} < \binom{2n}{k}$ by showing that $\binom{2n}{k-1} / \binom{2n}{k} < 1$.

3. Show that for any finite set that the number of subsets of even cardinality is equal to the number of subsets of odd cardinality.

4. Show that for all n
$$\sum_{i=0}^{n} \binom{n}{i}\binom{2n}{n+i} = \binom{3n}{n}$$

5. Show that for all n
$$\sum_{i=0}^{2n} \binom{2n}{i}\binom{3n}{n+i} = \binom{5n}{2n}$$

6. Find constants a, b, and c so that $i^2 = a\binom{i}{0} + b\binom{i}{1} + c\binom{i}{2}$. Use this expression and the Hockey Stick Identities to derive our formula for
$$\sum_{i=1}^{n} i^2$$

7. Find constants a, b, c, and d so that $i^3 = a\binom{i}{0} + b\binom{i}{1} + c\binom{i}{2} + d\binom{i}{3}$. Use this expression and the Hockey Stick Identities to derive our formula for
$$\sum_{i=1}^{n} i^3$$

8. Show that
$$\sum_{i=1}^{n} i(n-i+1) = \binom{n+2}{3}.$$

   Hint: The last three problems can be done by induction, or using the hockeystick identities, or by a combinatorial argument describing a particular method of creating subsets, for this problem creating 3 elements subsets from a set of cardinality $n+2$.

9. Show that
$$\sum_{i=0}^{n}(n+1-i)\binom{i+3}{3} = \binom{n+5}{5}.$$

10. Show that
$$\sum_{i=0}^{n}\binom{k+1}{2}\binom{n-k+1}{2} = \binom{n+4}{5}.$$