Lecture 14 Exercises

Like the problems we did in class this week, all these problems have solutions which involve defining a function. However, you have to think which functions might be useful. Sometimes the hint is that you are often trying to use the pigeonhole principle.

If you get stuck on a problem do not give up. The TA's have hints. Good luck.

1. Let P be the set of passengers on an airline. Define $Pocket(p)$ to be the amount of cash in the pocket of passenger p. If the security wants to use the amount of money to distinguish the passengers, what must be required of the function?

2. A polynesian king ruled 18 islands and had 10 sons. How many ways could he distribute the islands amongst his sons in his will, assuming that he is not so foolish as to divide any island.

   How many ways are there to distribute the islands so that each son can be king of at least one island.

3. Show that there are at least two U.S presidents who mother's maiden name had the last letter.

4. Let C be the set of characters in the play *Hamlet* and let $A = \{1, 2, 3, 4, 5\}$ be the set of acts. Define a function $F : C \longrightarrow \mathcal{P}(A)$ by letting $f(c)$ be the set of acts in which the character c appears.

   Show that this function is not onto. Is F one-to-one?

5. There is a party of 57 people. After the party starts, the doors are closed and nobody is allowed to enter or exit. Show at after 57 minutes, there are two persons at the party who have exchanged handshakes with the same number of people.

   Try to formulate a general statement.

6. Show that if you have five checkers on a checkerboard, there is at least one pair of checkers so that that there is a square halfway between them. What about a 3-d checkerboard?

7. Let A be a nonempty set of natural numbers. Prove that there exists a pair of numbers in A whose difference is divisible by $|A| - 1$.

8. The number 1111110 is divisible by 7 and 1111110000000 is divisible by 13. Show that for each number n there is a number m of the form $1 \ldots 0 \ldots$ which is divisible by n.

9. Consider the set $A = \{1, \ldots, 999\}$. Prove that any subset $B \subseteq A$ with $|B| = 501$ elements contains three numbers a, b, and c so that $c = a + b$.

10. Show that any subset of $\{1, 2, \ldots, 1000\}$ containing at least 501 elements has at least one pair of elements which are have no prime factor in common.

11. Show that any subset of $\{1, 2, \ldots, 1000\}$ containing at least 501 elements has at least one pair of elements one of which divides the other.

12. Show that every covering of a 6×6 square grid by 1×2 dominoes has at least one grid line, not along the border, which bisects no domino.

13. Ten points are placed in a unit square. Show that there are at least two points whose distance from one another is at most $1/3\sqrt{2}$.

14. Prove that an equilateral triangle cannot be covered by two strictly smaller equilateral triangles.