Exercises for Lectures 10 – Induction

1. Suppose that p_n is a statement which depends on the positive integer n. Suppose that p_5 is false, p_6 is false, p_7 is true, and $p_n \Longrightarrow p_{n+2}$.

   Does it follow that p_{10} is true?

   Does it follow that p_2 is false?

   What can you conclude about p_i using induction?

2. Suppose that p_n is a statement which depends on the positive integer n. Suppose that p_{50} is true, p_{500} is true, and $p_{n+1} \Longrightarrow p_n$.

   What can you conclude about p_i using induction?

3. Let p_n be the statement

   $$p_n := \qquad [n^2 - 5n \text{ is even.}]$$

   Prove p_n is true for all positive integers n by induction.

4. Let p_n be the statement

   $$p_n\colon \qquad 1 + 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}$$

   Prove p_n is true for all positive integers n by induction.

5. Prove by induction that $n^2 - n$ even for all integers $n > 0$.

6. Prove by induction that $n^3 - n$ is evenly divisible by 6 for all integers $n > 0$.

7. Prove by induction that $n^3 + 3n^2 + 2n$ is evenly divisible by 6 for all integers $n > 0$.

8. Prove by induction that $\sum_{i=1}^{n} i^3 = [n(n+1)/2]^2$.

9. Let $f(x) = \dfrac{1}{1+x}$. Show by induction that the sequence $f^{2n}(5)$ is decreasing.

10. Let $f(x) = \dfrac{1}{1+x}$. Show by induction that the sequence $f^{2n}(1)$ is increasing.

11. We showed that for sets that the union is associative

    $$A \cup (B \cup C) = (A \cup B) \cup C$$

    Use this result and induction to any union of n sets $\{A_1, \ldots, A_n\}$ can be parenthesized in any manner.

    Hint: Show any parenthesized union of $\{A_1, \ldots, A_n\}$ is equal to

    $$A_1 \cup (A_2 \cup (A_3 \cup \ldots \cup (A_{n-1} \cup A_n)))$$

12. We showed that for sets that the intersection distributes over the union

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Use this result and induction to show that for any set A and any finite collection of sets $\{B_1, \ldots, B_n\}$ we have

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$$

13. We showed that for sets that the intersection distributes over the union

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Use this result and induction to show that for any set two finite collections of sets $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_n\}$ we have

$$(A_1 \cup A_2 \cup \cdots \cup A_m) \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = \bigcup_{1 \le i \le m, 1 \le j \le n} (A_i \cap B_j)$$