CS5003 — Quiz 0101

Foundations of C.S.

Spring, 2020

PRINT NAME: ____________________

$\mathcal{SIGN}$: ____________________

1. Let A be the set of natural numbers which are defined recursively by
BASIS: 1 and 3 are in A.
RECURSIVE STEP: If $n \in A$, then $4n+1$ and $4n+3$ are both in A.
CLOSURE: $n \in A$ if a is obtained from the basis after a finite number of applications of the recursive step.

a) (**4** points) Write down all the elements in L_0 and L_2 .

♣ *This is the question I intended to ask:* L_0 *is the basis* $L_0 = \{1, 3\}$

L_1 *consists of* L_0 *together with* 5, 13, 7 *and* 15*:* $L_1 = 1, 3, 5, 7, 13, 15$

L_2 *consists of* L_1 *and new elements generated from* 5, 13, 7 *and* 15*, namely* $L_2 = 1, 3, 5, 7, 13, 15, 21, 23, 29, 31, 53, 55, 61, 63$. *Of course, in base* 4 *it is faster:* L_0 *is the basis*

$L_0 = \{1, 3\}$

$L_1 = \{1, 3, 11, 13, 31, 33\}$

$L_12 = \{1, 3, 11, 13, 31, 33, 111, 131, 311, 331, 113, 133, 313, 333\}$ ♣

Let $B \subseteq \mathbb{N}$ be the set of all natural numbers whose base four representation has only digits 1 and 3.

b) (**3** points) Show $A \subseteq B$

♣ *Let* $a \in A$. *We show* $a \in B$ *by induction on the number of recursive steps applied to generate* a.

Base Case: 0 *steps. Then* a *is in the basis, so* $a = 1$ *or* $a = 3$*, and* $a \in B$ *as required.*

Inductive Step: Suppose inductively that after n *steps we have generated* $a' \in B$*, and we apply the* $n+1$*'st step to generate* a. a' *is in* B*, so it is represented as a string of* 1*'s and* 3*'s. Multiplying by* e *shifts it by to the left and adds a zero at the end, and depending on which rule, we add a* 1 *or a* 3*. In either case,* $a \in B$*, as required.*

So $a \in B$ *no matter how many steps are required.*

(A less formal induction is ok, but for that you have to be even more careful with your argument!)

Here is a second version without appealing to strings of 1*'s and* 3*'s.*

Basis Case: The same

Inductive Step: Suppose inductively that after n *steps we have generated* $a' \in B$*, and we apply the* $n+1$*'st step to generate* a. a' *is in* B*,*

$$a' = f_0 4^0 + f_1 4^1 + \cdots + f_n 4^n, \qquad f_i \in \{1, 3\}$$

We will apply one rule, if the first, let $f = 1$ *and if the second, let* $f = 3$*. Then we get*

$$a' = f \cdot 4^0 + f_0 4^1 + f_1 4^2 + \cdots + f_n 4^{n+1}), \qquad f, f_i \in \{1, 3\}$$

so $a \in B$*, as required.* ♣

c) (**3** points) Show $B \subseteq A$

♣ *Again, I will put in two write-ups, one with base* 4 *lingo, and one without.*

Let $b \in B$. We have to show b can be recursively generated. and

$$b = f_0 4^0 + f_1 4^1 + \cdots + f_n 4^n \qquad f_i \in \{1, 3\}$$

So start with the basis, 1 is $f_n = 1$ and 3 if f_n is 3, and apply the recursive step n times, at the i'th step taking the first rule if $f_{n-i} = 1$, and the second if $f_{n-i} = 3$. This generates b, so $b \in A$.

Let $b \in B$. We have to show b can be recursively generated. b is represented by a a string of 1's and $3'$ of length n. So start with the basis, 1 if the leftmost digit is 1 and 3 otherwise, and apply the same criterion at the i'th step using depending on the i's digits from the left of n. This generates b, so $b \in A$.

♣