Ma2201/CS2022 Quiz 0100

Foundations of C.S.

Spring, 2019

PRINT NAME: __________________

SIGN: __________________

1. (**10 pts** Let L be the language defined recursively by

BASIS: $a \in L$

RECURSIVE STEP: $w \in L \Longrightarrow waa, wab, wba, wbb \in L$.

CLOSURE: Every element of L if obtained from the basis after a finite number of applications of the recursive step.

Let $M = \{w \in \{a,b\}^* \mid w = av, n_a(w) + n_b(w) = 2k+1, k \in \mathbb{N}\}$.

Prove that $L = M$.

HINT: You need to show $L \subseteq M$ and $M \subseteq L$ and both probably require an induction. Use the back if necessary.

♣ $L \subseteq M$. L is recursively defined, so we will show $L_n \subseteq M$ for all $n \geq 0$ by induction.

Base Case: $L_0 = \{a\}$ and $a \in M$ since $a = a\lambda$, $n_a(a) + n_b(a) = 1 + 0 = 1 = 2 \cdot 0 + 1$.

Inductive step. Suppose $L_n \subseteq M$. Let $x \in L_{n+1}$, and we can assume x is obtained from an element in L_n by the application of one rule, $x \in \{waa, wab, wba, wbb\}$ for some $w \in L_n$. Since $w \in M$, we have the first letter of w is a and $n_a(w) + n_b(w) = 2k+1$. So the first letter of x is a, and $n_a(x) + n_b(x) = n_a(w) + n_b(w) + 2 = 2k + 1 + 2 = 2(k+1) + 1$. So $x \in M$ as required, and $L_{n+1} \subseteq M$.

$M \subseteq L$. We have to show that every $w \in M$ is L. So the first letter of w is a and $n_a(w) + n_b(w) = 2k+1$. We do this by induction on k.

Base Case: $k = 0$. Then $w = a$, and $w \in L$ by the basis of L.

Inductive step: Suppose every word of length $2k+1$ which starts with a is in L. Let w start with a and be of length $2(k+1)+1 = 2k+3$. The $w = avpq$, where the length of $p, q \in \{a,b\}$ and $n_a(av) + n_b(av) = 2k+1$. So $av \in L$ by the inductive hypothesis, and there is a rule of the recursively defined set which implies $avpq \in L$, as required.

So every w in L for all $k \geq 0$ by induction. ♣