CS5003
Quiz 0010

Foundations of C.S.

Spring, 2017

PRINT NAME: ____________________

SIGN: ____________________

1. (4 points) Give a recursive definition of the set

$$X = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid m > 5\} \cup \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid n > 7\}.$$

♣

BASIS: $(0, 6) \in X$ and $(8, 0) \in X$

RECURSIVE STEP: If $(n, m) \in X$ then $(n, m + 1) \in X$ and $(n + 1, m) \in X$.

CLOSURE: All elements in X are obtained from a basis element after a finite number of applications of the recursive step.♣

(You were not asked to prove your construction was correct, but I could have asked that too. Here is how it would go.)

Let Y be the set defined by the recursive definition. We want to show $X = Y$.

To show $X \subseteq Y$. Let $x \in X$, so $x = (n, m)$ with $n > 7$ or $m > 5$. If $m > 5$, then $m - 6 \geq 0$. The pair $(0, 6) \in Y$ is an a basis element, and $m - 6$ applications of the recursive step shows $(0, 6 + (m - 6)) = (0, m) \in Y$, so and n applications of the other rule of the recursive step shows $(0 + n, m) = (n, m) = x \in Y$, as required.

(The case $n > 7$ is similar, you can try it yourself.)

To show $Y \subseteq X$. Let $y \in Y$. $y = (n, m)$. We need induction on the number of recursive steps in the construction of y.

Base Case: 0 steps, so $y = (0, 6)$, with $6 > 5$, so $y \in X$, or $y = (8, 0)$ with $8 > 7$, so again $y \in X$.

Induction Step: $n > 0$ steps. The elements $y = (n, m)$ is constructed either from $y' = (n - 1, m)$ or $y'' = (n, m - 1)$, so either y' or y'' is in X.

If $y' \in X$, then either $n - 1 > 7$ or $m > 5$. If $m > 5$ then $y \in X$ as well. If $n - 1 > 8$ then $n > 9 > 8$ and again $y \in X$, as required.

On the other hand,if $y'' \in X$, then either $n > 7$ or $m - 1 > 5$. If $n > 7$, then $y \in X$ as well. If $m - 1 > 5$, then $m > 6 > 5$, so again $y \in X$, as required.

So in either case, $y \in X$, and the induction step in shown.

Since the base case and the induction step have been show, $y \in X$ for all $y \in Y$ by induction.

Since $X \subseteq Y$ and $Y \subseteq X$, $X = Y$, and the recursive definition describes X.

2. (3 points) Prove whether or not the set relation

$$R = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n = m + 7k, \quad k \in \mathbb{Z}\}$$

is an equivalence relation.

♣ We have to prove R is reflexive, symmetric, and transitive.

Reflexive: Let $n \in \mathbb{Z}$. $n = n + 0k$, so $(n, n) \in R$, so $(n, n) \in R$ for all n, and R is reflexive.

Symmetric: Let $(n, m) \in R$, so $n = m + 7k$ with $k \in \mathbb{Z}$. Then $m = n - 7k$, or $m = n + 7(-k)$, and $-k \in \mathbb{Z}$, so $(m, n) \in R$. Thus R is symmetric.

Transitive: Let $(n, m) \in R$ and $(m, i) \in R$. Then $n = m + 7k$ and $m = i + 7k'$, for two values $k, k' \in \mathbb{Z}$. So $n = (i + 7k') + 7k = i + 7(k' + k)$, and $k' + k \in \mathbb{Z}$, so $(n, i) \in R$. So R is transitive. ♣

3. (3 pts) Label each of the following TRUE or FALSE.

___ $\bigcup_{k=2}^{\infty} \{(n, n + k^2) \mid n \in \mathbb{N}\}$ is countable.

♣ TRUE: Every element in this set is in the countable set $\mathbb{N} \times \mathbb{N}$, and a subset of a countable set is countable.

___ If A and B are both countably infinite, then $|A| = |B|$.

♣ TRUE: Countable says $|A| \leq |\mathbb{N}|$, so either A is finite or $|A| = |\mathbb{N}|$. A and B countably infinite says $|A| = |\mathbb{N}|$ and $|B| = |\mathbb{N}|$ so $|A| = |B|$.

___ If A is countably infinite, then $A \subseteq \mathbb{N}$.

♣ FALSE: $\mathbb{N} \times \mathbb{N}$ is countably infinite, but $\mathbb{N} \times \mathbb{N} \not\subseteq \mathbb{N}$.