
CS5003
Final Exam

# Foundations of C.S.

Spring, 2015

PRINT NAME: _____
$\mathcal{SIGN}$: _____

**Do any six problems. Write your answers clearly and neatly. Use the back if necessary.**

1. Let $L$ be the language with definition

$$L = \{a^i b^k \mid 0 \leq i \leq k \leq 2i\}.$$

Give a recursive definition of $L$. Find $L_3$ according to your definition. Finally give a context free grammar which realizes $L$.

Solution:
BASIS: $\lambda \in L$
RECURSIVE STEP: $u \in L \Rightarrow aub, aub^2 \in L$.
CLOSURE: All elements in the language can be obtained from the basis with a finite number of applications of the recursive step.

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With this definition:
$L_0 = \{\lambda\}$.
$L_1 = \{\lambda, ab, ab^2\}$.
$L_2 = \{\lambda, ab, ab^2, a^2b^2, a^2b^3, a^2b^4\}$.
$L_3 = \{\lambda, ab, ab^2, a^2b^2, a^2b^3, a^2b^4, a^3b^3, a^3b^4, a^3b^5, a^3b^6\}$.

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$$G : S \Rightarrow aSb \mid aSb^2$$

2. Let $L$ be given by

BASIS: $b, b^2, b^3 \in L$

RECURSIVE STEP: If $u \in L$ then $auc \in L$ and $babuc^3 \in L_2$.

CLOSURE: A string is in $L$ if if can be obtained from the basis by a finite number of applications of the recursive step.

Prove by induction that for all $w \in L$, $n_a(w) + n_c(w)$ is even.

SOLUTION:

Proof. We will prove the statement by induction on the number of steps in the recursive definition of $w \in L$.

Base Case: Suppose $w \in L_0$. Then $w \in \{b, b^2, b^3\}$, so $n_a(w) = n_c(w) = 0$, and $n_a(w) + n_c(w) = 0$ which is even.

Inductive step. Suppose $n_a(u) + n_c(u)$ is even for all $u \in L_n$. Let $w \in L_{n+1}$, so either $w = auc$ or $w = auc^3$.

If $w = auc$ then $n_a(w) + n_c(w) = n_a(u) + 1 + n_c(u) + 1 = (n_a(u) + n_c(u)) + 2$ which is even by the induction hypothesis.

If $w = auc^3$ then $n_a(w) + n_c(w) = n_a(u) + 1 + n_c(u) + 3 = (n_a(u) + n_c(u)) + 4$ which is even by the induction hypothesis.

In either case $n_a(w) + n_c(w)$ is even, which completes the induction step.

So the statement is proved by induction.

3. Give regular expressions for each of the following subsets of $\{a, b, c\}^*$.

a) Strings which do not contain the substrings $aa$ or $bb$.

SOLUTION:

Without $c$ as a separator you have even or odd alternating $a$'s and $b$'s, so you could take $[(ab)^*(a \cup \lambda)] \cup [(ba)^*(b \cup \lambda)]$.

With $c$, you can take

$$c^*\{[[(ab)^*(a \cup \lambda)] \cup [(ba)^*(b \cup \lambda)]]c^+\}^*[[(ab)^*(a \cup \lambda)] \cup [(ba)^*(b \cup \lambda)]]$$

Note this also matches the empty string.

b) Set of strings of odd length containing exactly two $b$'s.

SOLUTION: It seems simplest to think of four cases, where the three strings separate by the two $b$'s have length even-even-odd, even-odd-even, odd-even-even, or odd-odd-odd

$$\begin{aligned}
&[((a \cup c)^2)^*]b[((a \cup c)^2)^*]b[((a \cup c)^2)^*(a \cup c)] \\
\cup&[((a \cup c)^2)^*]b[((a \cup c)^2)^*(a \cup c)]b[((a \cup c)^2)^*] \\
\cup&[((a \cup c)^2)^*(a \cup c)]b[((a \cup c)^2)^*]b[((a \cup c)^2)^*] \\
\cup&[((a \cup c)^2)^*(a \cup c)]b[((a \cup c)^2)^*(a \cup c)]b[((a \cup c)^2)^*(a \cup c)]
\end{aligned}$$

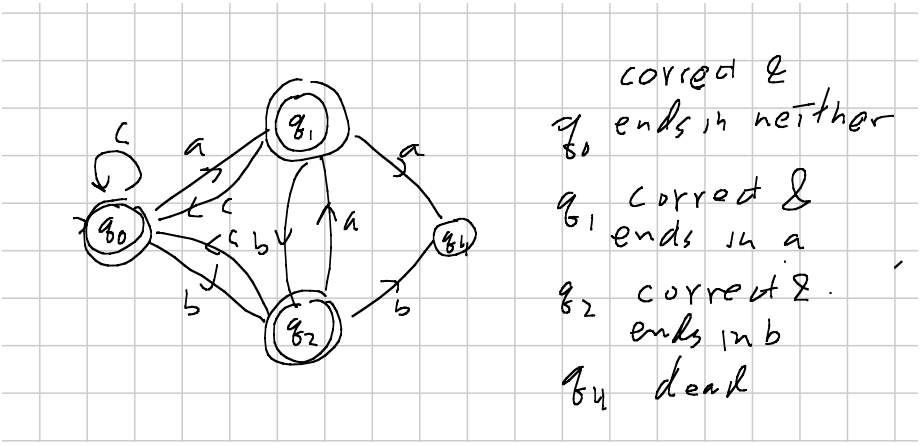
c) Set of strings with an even number of $a$'s

SOLUTION:

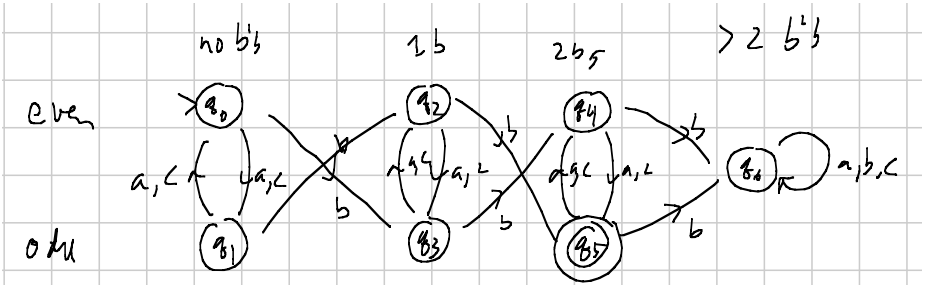
$$[(a(b \cup c)^*a) \cup b \cup c]^*$$

4. Give a deterministic finite automaton which realizes each of the following languages on $\{a, b, c\}$.

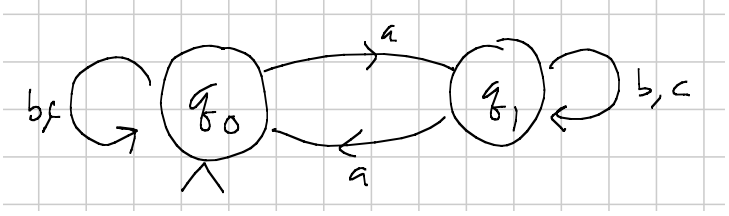
a) Strings which do not contain the substrings $aa$ or $bb$.




b) Set of strings of odd length containing exactly two $b$'s.




c) Set of strings with an even number of $a$'s



5. Construct a context free grammar whose language is $L = \{a^m b^i a^n \mid i = m + n\}$.
Prove that your grammar is correct.

SOLUTION:

$$\begin{aligned} G : S &\rightarrow AB \\ A &\rightarrow aAb \mid \lambda \\ B &\rightarrow bBa \mid \lambda \end{aligned}$$

We will $S(G)$ the sentential forms of the grammar, is equal to the set
$X = \{S, a^m Ab^m b^n Ba^n, a^m Ab^m b^n a^n, a^m b^m b^n Ba^n, a^m b^m b^n a^n\}$ with $n, m \geq 0$.

Let $S$ Proof: $S(G) \subseteq X$. Induction on the number of rules applied.

Base Case: If 0 rules have been applied, the sentential form is $S$, which is in $X$ as required.

Inductive Step. Assume that, after $n$ rules have been applied, the sentential form $n$ is contained in $X$. If a rule applies $w$ has an $S$, and $A$ or a $B$.

If $w$ has an $S$, then $w = S$ and the applying the only rule gives $AB \in X$.

If an $A$ rule applies then $w$ is $a^m Ab^m b^n Ba^n$ or $a^m Ab^m b^n a^n$ and applying the rule gives $a^{m+1} Ab^{m+1} b^n Ba^n$, $a^{m+1} Ab^{m+1} b^n a^n$, $a^m b^m b^n Ba^n$ or $a^m b^m b^n a^n$, each of which are in $X$.

If a $B$ rule applies then $w$ is $a^m Ab^m b^n Ba^n$ or $a^m b^m b^n Ba^n$ and applying the rule gives $a^m Ab^m b^{n+1} Ba^{n+1}$, $a^m b^m b^{n+1} Ba^{n+1}$, $a^m Ab^m b^n a^n$, or $a^m b^m b^n a^n$, each of which are in $X$,

So in any case, applying a rule gives a new rule in $X$ and by induction $L(G) \subseteq X$

To show $X \subseteq L(G)$, we provide a derivation sequence for the elements of $X$.

$$S \Rightarrow AB \stackrel{m}{\Rightarrow} a^m Ab^m B \stackrel{n}{\Rightarrow} a^m Ab^m b^n Ba^m \Rightarrow a^m b^m b^n Ba^m \Rightarrow a^m b^m b^n a^m$$

Are derivation sequences for $a^m Ab^m b^n Ba^m$, $a^m b^m b^n Ba^m$ and $a^m b^m b^n a^m$ and

$$S \Rightarrow AB \stackrel{m}{\Rightarrow} a^m Ab^m B \stackrel{n}{\Rightarrow} a^m Ab^m b^n Ba^m \Rightarrow a^m Ab^m b^n a^m$$

are is a derivation sequence for $a^m Ab^m b^n a^m$.

So $X$ is the set of sentential forms, and the the elements with no variables are the language $L$, so $L = L(G)$.

6. Let $G$ be the grammar given by

$$\begin{aligned} G : S &\to aAbB \mid ABC \mid a \\ A &\to aA \mid a \\ B &\to bBcC \mid b \\ C &\to abc \\ D &\to aAbBcCCC \mid AaA \mid E \\ E &\to DE \mid ED \end{aligned}$$

Convert this grammar to Chomsky Normal Form.

First note that the final two symbols are useless, and can be eliminated. $D$ is unreachable and $E$ is non-terminable.

The grammar is non-contracting, and removing $D$ removes the only chain rule. If you keep $D$ and $E$ this chain rule must be removed. We will follow the procedure we covered in class: We first introduce $C_a$, $C_b$, $C_c$

$$\begin{aligned} G : S &\to C_aAC_bB \mid ABC \mid a \\ A &\to C_aA \mid a \\ B &\to C_bBC_cC \mid b \\ C &\to C_aC_bC_c \\ C_a &\to a \\ C_b &\to b \\ C_c &\to c \end{aligned}$$

and then split all the long rules:

$$\begin{aligned} G : S &\to C_aF_1 \mid AG_1 \mid a \\ A &\to C_aA \mid a \\ B &\to C_bH_1 \mid b \\ C &\to C_aG_1 \\ F_1 &\to AF_2 \\ F_2 &\to C_bB \\ G_1 &\to BC \\ H_1 &\to BH_2 \\ H_2 &\to C_cC \\ G_1 &\to C_bC_c \\ C_a &\to a \\ C_b &\to b \\ C_c &\to c \end{aligned}$$

7. a) Let $G$ be the grammar given by

$$\begin{aligned} G: S &\rightarrow AB \mid C \\ A &\rightarrow aA \mid B \\ B &\rightarrow bB \mid C \\ C &\rightarrow cC \mid A \mid a \end{aligned}$$

Construct an equivalent grammar which does not contain chain rules.

We have seen this silly example before. $CHAIN(S) = \{S, A, B, C\}$, $CHAIN(A) = \{A, B, C\}$, $CHAIN(B) = \{A, B, C\}$, $CHAIN(C) = \{A, B, C\}$,

So $A$, $B$, and $C$ are all equivalent.

We can either simplify to

$$\begin{aligned} G: S &\rightarrow AB \mid aA \mid aB \mid aC \mid a \\ A &\rightarrow aA \mid aB \mid aC \mid a \\ B &\rightarrow aA \mid aB \mid aC \mid a \\ C &\rightarrow aA \mid aB \mid aC \mid a \end{aligned}$$

or just

$$\begin{aligned} G: S &\rightarrow AA \mid aA \mid a \\ A &\rightarrow aA \mid a \end{aligned}$$

b) For the grammar

$$\begin{aligned} G: S &\rightarrow AB \mid BCS \\ A &\rightarrow aA \mid C \\ B &\rightarrow bbB \mid b \\ C &\rightarrow cC \mid \lambda \end{aligned}$$

construct an essential non-contracting grammer $G_L$ with non-recursive start

For the non-recursive start we "step back" to a new start symbol $S'$, and introduce a new chain rule. (The directions to require us to remove the chain rules.)

$$\begin{aligned} G: S' \quad & S \\ S &\rightarrow AB \mid BCS \\ A &\rightarrow aA \mid C \\ B &\rightarrow bbB \mid b \\ C &\rightarrow cC \mid \lambda \end{aligned}$$

$Null(G) = \{A, C\}$ so we translate by

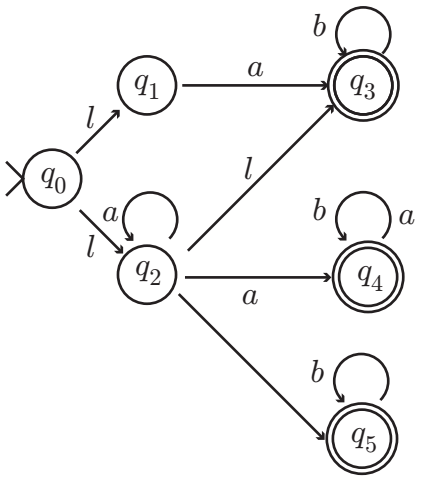
$$\begin{array}{rcl} G : S' & S & \\ S & \rightarrow & AB \mid BCS \mid B \mid |BS \\ A & \rightarrow & aA \mid C \mid a \\ B & \rightarrow & bbB \mid b \\ C & \rightarrow & cC \mid c \end{array}$$

If you also remove the chain rules, $CHAIN(S') = \{S', S, B\}$, $CHAIN(S) = \{S, B\}$, $CHAIN(A) = \{A, C\}$, $CHAIN(B) = \{B\}$, and $CHAIN(C) = \{C\}$. and we get

$$\begin{array}{rcl} G : S' \quad AB \mid BCS \mid bbB \mid b \mid |BS & & \\ S & \rightarrow & AB \mid BCS \mid bbB \mid b \mid |BS \\ A & \rightarrow & aA \mid cC \mid c \mid a \\ B & \rightarrow & bbB \mid b \\ C & \rightarrow & cC \mid c \end{array}$$

Both the pervious two grammars solve the problem. And any essentially non-contracting grammar giving the same language does as well, since the question did not specify how it was to be created.

8. Find an equivalent Deterministic Finite Automaton. (Note that in the diagram, $\lambda$ is denoted by $l$.)

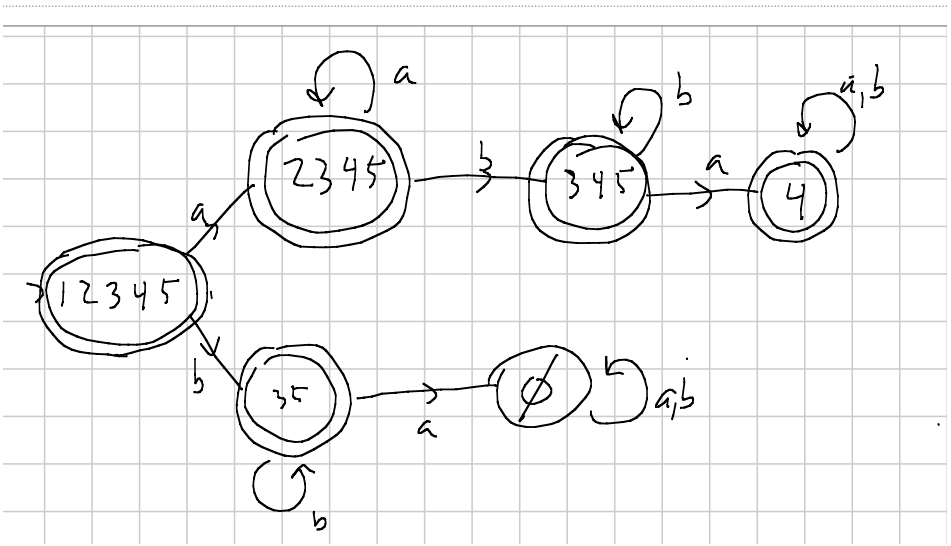



Also give a regular expression for the language.

We first compute the transition function:

| | $\lambda-$ closure | $a$ | $b$ |
|---|---|---|---|
| $q_0$ | $\{q_0, q_1, q_2, q_3, q_5\}$ | $\{q_2, q_3, q_4, q_5\}$ | $\{q_3, q_5\}$ |
| $q_1$ | $\{q_1\}$ | $\{q_3\}$ | $\emptyset$ |
| $q_2$ | $\{q_2, q_3, q_5\}$ | $\{q_2, q_3, q_4, q_5\}$ | $\{q_3, q_5\}$ |
| $q_3$ | $\{q_3\}$ | $\emptyset$ | $\{q_3\}$ |
| $q_4$ | $\{q_4\}$ | $\{q_4\}$ | $\{q_4\}$ |
| $q_5$ | $\{q_5\}$ | $\emptyset$ | $\{q_5\}$ |

Giving the automaton.




For the regular expression we can use the expression graphs, and the regular expression easily simplifies to $b^+ \cup a(a \cup b)^*$.