Summary of Lectures 21 and 22

We talked about predicate logic and induction.

If $X_1, \ldots X_k$ are sets, we can define a Boolean function

$$p : X_1 \times \cdots \times X_k \to \{0, 1\}$$

and $p = p(x_1, \ldots, x_k)$ is called a *predicate* whose truth depends on how $x_1, \ldots, x_k$ are *quantified*. (This is just logical jargon, the mathematical notations and concepts are still valid and unchanged.)

Example: $p(n, m) =$ "$n^2 + m$ is prime" is a predicate, with $p(3, 1)$ false and $p(5, 4)$ true.

Besides direct quantification of the predicate (evaluation of the function) we talked of *universal* and *existential* quantification.

Universal: $\forall\ n \in \mathbb{Z}, n^2 \geq 0$. (TRUE)

Existential: $\exists\ n \in \mathbb{N}, n^2 + 17$ is prime. (FALSE)

We showed how the order of quantifying predicates which depend on several variables matters. We also described how to negate universally and existentially.

We talked about proving statements universally quantified over the natural numbers *by induction*

$$[\forall\ n \in \mathbb{N}; p_n] \iff \overbrace{p_0}^{\text{base case}} \wedge \overbrace{[\forall\ n \in \mathbb{N}; p_n \Rightarrow p_{n+1}]}^{\text{induction step}}$$

As an example we proved

$$\forall\ n \in \mathbb{N}; \exists\ k \in \mathbb{N}; n^3 - 11n = 6k.$$

Exercises for Lectures 21 and 22

1. For each of the following statements, express without words using sets and logical notation.
   (a) The natural numbers n and m sum to less than their square.
   (b) The natural numbers x and y satisfy $y = x^2$.
   (c) For every natural number there is a power of 10 that exceeds it.
   (d) For every natural number there is a power of 10 that exceeds it.
   (e) The set of squares of integers is unbounded.
   (f) The square root of two exists. (try not to use the square root symbol)
   (g) Every non-negative real number has a square root. (try not to use the square root symbol)

2. For each of the following statements, express as simply as possible in words. They are all either true or unquantified.

   (a) $n^2 + m^2 = 1000$
   (b) $\forall\ n \in \mathbb{Z}; 2^n > n$
   (c) $\forall\ n \in \mathbb{Z}; 3^n \geq 2^n$
   (d) $\exists\ n \in \mathbb{Z}; n > 1000$
   (e) $\exists\ x \in \mathbb{R}; x^5 = 7$
   (f) $\forall\ a \in \mathbb{R}; \exists\ b \in \mathbb{R}; b^5 = a$
   (g) $\forall\ a \in \mathbb{R}; \exists\ b \in \mathbb{R}; a^5 = b$

3. Give the negation of each of the following predicates.

   (a) $p(n, m) \wedge p(n + 1, m + 1)$
   (b) $\forall\ n; p(n, m) \wedge (p(n + 1, m + 1) \vee p(n, m - 1))$
   (c) $\forall\ n \in \mathbb{N}; \exists\ m \in \mathbb{Z}; p_{n,m}$
   (d) $\forall\ n \in \mathbb{Z}; 2^n > n$
   (e) $\forall\ n \in \mathbb{Z}; 3^n \geq 2^n$
   (f) $\exists\ n \in \mathbb{Z}; n > 1000$
   (g) $\exists\ x \in \mathbb{R}; x^5 = 7$
   (h) $\forall\ a \in \mathbb{R}; \exists\ b \in \mathbb{R}; b^5 = a$
   (i) $\forall\ a \in \mathbb{R}; \exists\ b \in \mathbb{R}; a^5 = b$

4. Suppose p_n is a predicate with $n \in \mathbb{N}$. Suppose p_0 is true and

$$\forall\ n \in \mathbb{N}; (p_n \Rightarrow p_{n+2}) \wedge (p_n \Rightarrow p_{n+5})$$

   Find all that you can determine by induction.

   Just for fun: Find a predicate which satisfies this and is *not* true for all n.

   ♣ *Since p_0 is true, $p_n \Rightarrow p_{n+2}$ and induction implies that p_k is true whenever k is even. Also, since p_0 is true, $p_n \Rightarrow p_{n+5}$ gives p_5 is true and by induction, and p_k whenever k is a multiple of 5. Moreover, since p_5 is true, $p_n \Rightarrow p_{n+2}$ and induction imply that p_k is true for all odd numbers 5 and beyond. So the only values left which could be false are p_1 and p_3.*

   For those, any combination works except $p_1 = 1$ and $p_3 = 0$. ♣

5. Suppose p_n is a predicate with $n \in \mathbb{N}$. Suppose p_{25} is false and

$$\forall\ n \in \mathbb{N}; (p_n \Rightarrow p_{n+2}) \wedge (p_n \Rightarrow p_{n+5})$$

   Must p_1 be true?

   [*Note: Don't try these next proofs all in one day. Pace them out. You will learn the method much better that way.*]

6. Prove by induction that $n^3 - n$ is always divisible by 3.
7. Prove by induction on n that $\sum_{k=1}^{n} k = (n)(n+1)/2$.

   Can you also prove it by induction on k? Why or why not?
8. Prove by induction on n that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

9. Prove by induction on n that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

10. Prove by induction on n that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

11. Prove by induction on n that

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$$

    Careful – what is the base case?
12. Prove by induction on n that

$$\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \cdots + \binom{n}{4} = \binom{n+1}{5}$$

    Careful – what is the base case?
13. No problem to do, just a illustration of the intersection of logic, negation, and natural language:

    THE MARK: Ok, I'm here.
    BUD ABBOTT: Who says you're here?
    THE MARK: What do you mean? I'm here. Of course, I'm here!
    BUD ABBOTT: You wouldn't want to bet on it?
    THE MARK: You want to bet me that I'm – not here? Are are you crazy or something?
    BUD ABBOTT: Right! I'll bet you you're not here! Ten dollars says you're not here!
    THE MARK: Well, that's a bet, chump. Come on.
    BUD ABBOTT: Here. You can hold it too.
    THE MARK: (Taking the two 10 spots.) Ten dollars ...
    BUD ABBOTT: That you're not here!

The Mark: Alright, alright, prove it – that I'm not here.
Bud Abbott: You're not in Chicago, are you?
The Mark: Why, certainly not!
Bud Abbott: No ... you're not in Philadelphia, are you?
The Mark: No!
Bud Abbott: No ... You're not in Saint Louis, are you?
The Mark: Course not!
Bud Abbott: Well, wait a moment ... you're not in Chicago, you're not in Philadelphia, and you're not in Saint Louis, you must be someplace else!
The Mark: That's right!
Bud Abbott: Well, if you're someplace else, you can't be here!
The Mark: Right! ... No! ...
Bud Abbott: [Snatching the money in triumph] There you go!