Summary of Lectures 19 and 20

We defined and gave examples of *formal Boolean functions* (binary functions)

$$b : \{0,1\}^n \to \{0,1\}$$

We showed that every formal Boolean function can be written in terms of its logical variables in terms of the operators $\vee$, $\wedge$ and $\neg$.

More precisely we showed that every formal Boolean function can be written in *disjunctive normal* – an OR statement of only AND clauses.

Example:

$$(p \wedge q \wedge \neg r) \vee (\neg q) \vee (\neg p \wedge q \wedge r)$$

Dually, it can be written in *conjunctive normal form* – an AND statement of only OR clauses.

Example:

$$(p \vee q \vee \neg r) \wedge (\neg q) \wedge (\neg p \vee q \vee r)$$

We compared the two forms and gave examples of each.

We discussed logical implication: $p \Rightarrow q$, IF p THEN q, or p IMPLIES q.

$$(p \Rightarrow q) = (q \vee \neg p)$$

IMPORTANT: We showed the double implication method for showing $p \Leftrightarrow q$ by showing separately that $p \Rightarrow q$ and $q \Rightarrow p$

Exercises for Lectures 19 and 20

1. Prove that are $2^{(2^2)} = 16$ Boolean functions on the two Boolean variables p and q.

   One is $p \vee (q \wedge \neg p)$

   Find as many as you can.

   Hint: They can all be expressed as the one above, in terms of AND, OR and NOT.

2. Write the Boolean function $p \Rightarrow (q \Rightarrow r)$ in terms of AND, OR and NOT.

3. Write the Boolean function $(p \Rightarrow q) \Rightarrow r$ in terms of AND, OR and NOT.

4. Write the Boolean function $(p \Rightarrow q) \Leftrightarrow (r \Rightarrow q)$ in terms of AND, OR and NOT.

5. Write a Boolean function on the variables p, q and r with the property that it returns TRUE precisely when at least two of the three variables p, q and r are TRUE.

   ♣ *It is at least two out of three, so we have to look at each of the* $\binom{3}{2} = 3$ *possibilities and write a clause for each one of them.*

So $(p \wedge q)$ has p and q true, leaving r unspecified. For all 3 we get

$$(p \wedge q) \vee (p \wedge r) \vee (q \wedge r) \quad \clubsuit$$

6. Write a Boolean function on the variables p, q and r with the property that it returns TRUE precisely when at most two of the three variables p, q and r are TRUE.
7. For $p \Rightarrow (q \Rightarrow r)$ give an expression in conjunctive normal form.
8. For $p \Rightarrow (q \Rightarrow r)$, give an expression in disjunctive normal form.
9. For $(p \Rightarrow q) \Rightarrow r$ give an expression in conjunctive normal form.

$\clubsuit$ *Best to first decode the implications using the definition:*

$$[(p \Rightarrow q) \Rightarrow r] = r \vee \neg(p \Rightarrow q) = r \vee \neg(q \vee \neg p) = r \vee (\neg q \wedge p)$$

where the last uses Demorgan's law.

So we have an expression in terms of the desired operations, but it is in disjunctive normal form, not conjunctive. To fix that, we can just use the distributive law:

$$r \vee (\neg q \wedge p) = (r \vee \neg q) \wedge (r \vee p)$$

which is the statement in conjunctive normal form. $\clubsuit$

10. For $(p \Rightarrow q) \Rightarrow r$, give an expression in disjunctive normal form.
11. Suppose we have the implication "If Bob is a fireman, then Bob hates birthdays." Write down the inverse, converse, and contrapositive statements.
12. For $p \Rightarrow (q \Rightarrow r)$, write down and label the inverse, converse, and contrapositive statements.
13. For $(p \Rightarrow q) \Rightarrow r$, write down and label the inverse, converse, and contrapositive statements.