Summary of Lectures 15 and 16

Relation from a set to itself, $R \subseteq X \times X$.

We discussed properties of such relations, specfically

- Reflexivity and Anti-reflexivity
- Symmetry and Anti-symmetry
- Transitivity

We gave several examples.

A relation which is reflexive, symmetric, and transitive is called an equivalence relation.

An equivalence relation partitions a set into equivalence classes.

A partition of a set X is a set of subsets $X_1, \ldots, X_k$, $X_i \subseteq X$ satisfying $X = \cup_{i=1}^{k} X_i$ and $X_i \cap X_k = \emptyset$ for all $1 \leq i < j \leq k$.

We showed that if $f : A \to B$ is a function, then setting aRa' if $f(a) = f(a')$ defines an equivalence relation on A in which the equivalence classes partition the domain as the non-empty subsets of the form $f^{-1}(b) = \{a \in a \mid f(a) = b\}$.

Exercises

1. Draw the relation diagram of a relation on $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ which is antisymmetric and transitive.

2. Draw the relation diagram of the relation on $\mathcal{P}(\{a, b, c\})$ defined by A is related to B is $A \subseteq B$.

   What properties does this relation have?

3. Consider a relation $\blacktriangle$ on $\{0, 1, 2, 3, 4, 5\}$ and suppose $0\blacktriangle1$, $1\blacktriangle2$, $2\blacktriangle1$, $2\blacktriangle3$, $3\blacktriangle4$, and $4\blacktriangle5$.

   Draw the relation diagram draw the arrows corresponding these relationships, and all the consequent ones if $\blacktriangle$ is transitive.

4. Consider a relation $\bigstar$ on $\{0, 1, 2, 3, 4, 5\}$ and suppose $0\bigstar1$, $1\bigstar2$, $2\bigstar1$, $2\bigstar3$, $3\bigstar4$, and $4\bigstar5$.

   Draw the relation diagram draw the arrows corresponding these relationships, and all the consequent ones if $\bigstar$ is symmetric.

5. Consider a relation $\square$ on $\{0, 1, 2, 3, 4, 5\}$ and suppose $0\square1$, $1\square2$, $2\square1$, $2\square3$, $3\square4$, and $4\square5$.

   Draw the relation diagram draw the arrows corresponding these relationships, and all the consequent ones if X is symmetric and transitive.

6. Consider a relation on the set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ defined by setting $f \Diamond g$ if $f'(x) = g'(x)$.

   Is $\Diamond$ and equivalence relation?

   If not, which of the three qualities is violated, and for what functions?

   If so, find the equivalence class of $f(x) = x^2$.

7. Let $W = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0\}$. (So $(5, 3)$, $(2, -1) \in \mathbb{W}$. Define a relation $L \subset W \times W$ by

   $$L = \{(n, m), (n'm')) \mid nm' = mn'\}$$

   a) Show that W is an equivalence relation. (Check carefully all three required qualities.)

   b) Describe at least three equivalence classes.

   c) The elements of W correspond to integer points in the plane. Draw a grid at least 10×10 and mark in it, in three different colors, all the points belonging to three different equivalence classes, say that of $(1, 1)$, $(0, 1)$ and $(1, 2)$.

   c) (Just for fun.) Can you give a geometric description of the equivalence classes?

   d) (Just for fun.) Can you give any reason why the guy who defined L didn't want any W to have points like $(3, 0)$ or $(0, 0)$.

   e) (Just for fun.) Can you find a function $f : W \to \mathbb{Q}$ whose domain partition gives the same equivalence relation?

8. Let DS be the set of all words occurring in Dr. Seuss books of with 10 or fewer characters. Defined a relation R on DS by setting wRw' if w and w' rhyme in spoken English.

   Label each of the following T for TRUE, F for false, or R if you refuse to do such a silly problem.

   ____ The relation DS is anti-transitive.

   ____ The relation DS is anti-symmetric

   ____ The relation DS is anti-transitive.

   ____ The relation DS is antediluvian .

   ____ *box*, *socks*, and *fox* belong to the same equivalence class.

   ____ The equivalence class of *the* is empty.