Summary of Lectures 13 and 14

Countability, and cardinality and functions on infinite sets.

Discrete cantor's diagonal argument to give another proof that $|\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ are uncountable sets.

Applied the notation that

— a one-to-one function $A \to B$ gives $|A| \le |B|$, and

— an onto function $A \to B$ gives $|B| \le |A|$

to show $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are all countable infinite, that is $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$.

Showed that if A and B are countably infinite

— Any subset of A or B is finite or countably infinite.

— $A \cap B$ is finite or countably infinite.

— $A \cup B$ is countably infinite.

— $A \times B$ is countably infinite.

— $\mathcal{P}(A)$ is uncountable, but $\mathcal{P}_k(A)$ is countable for any $k \in \mathbb{N}$.

— A countable union of countable sets is countable.

— A countable product of sets containing at least two elements is uncountable.

And noted the principle that, any set which contains a subset which is encoded by an infinite number of non-trivial independent choices, is uncountable.

Exercises

For the following, you can use functions to show directly, or any result we showed in class.

1. Show $\mathcal{P}_2(\mathbb{Q})$ is countable. Find 5 elements in the set.
2. Show $\mathcal{P}_2(\mathcal{P}_2(\mathbb{N}))$ is countable. Find 5 elements in the set.
3. Decide if $\mathcal{P}_2(\mathbb{N}) \times \mathcal{P}_3(\mathbb{N})$ is finite, countably infinite, or uncountable. Find 5 elements in the set.
4. Decide if $\mathcal{P}(\mathcal{P}_2(\mathbb{N})) \times \mathcal{P}_3(\mathcal{P}(\mathbb{N}))$ is finite, countably infinite, or uncountable. Find 5 elements in the set.
5. Decide if $\mathcal{P}(\mathcal{P}_2(\mathbb{N})) \cup \mathcal{P}_3(\mathcal{P}(\mathbb{N}))$ is finite, countably infinite, or uncountable. Find 5 elements in the set.

   ♣ *The set $\mathcal{P}_2(\mathbb{N})$ of 2-subsets of $\mathbb{N}$ is countable. You can say it is the union of a countable collection of countable sets, in this case $\cup_{n=0}^{\infty}\mathcal{P}(\{0, 1, 2, 3, \ldots, n\})$, or you can just consider the onto function $f : \mathbb{N} \times \mathbb{N} \to \mathcal{P}_2(\mathbb{N})$ with countable domain set defined by $f(n, m) = \{n, m\}$ if $n \ne m$ and $f(n, n) = \{1, 2\}$. Then the power set of a countable set is uncountable, and the union of an uncountable set of anything is uncountable.*

   But you can also do it even you are unsure if $\mathcal{P}_2(\mathbb{N})$ is countable. It is certainly true that $\mathcal{P}_2(\mathbb{N})$ is infinite, and our result that $|\mathcal{P}(\mathcal{P}_2(\mathbb{N}))| > |\mathcal{P}_2(\mathbb{N})|$, says that the larger set is definitely uncountable. ♣

6. Decide if $\mathcal{P}(\mathcal{P}_2(\mathbb{N}))\cap\mathcal{P}_3(\mathcal{P}(\mathbb{N}))$ is finite, countably infinite, or uncountable. Find 5 elements in the set.

7. Show that the set of reals numbers whose base 10 representation has no 0 or 9 is uncountable.

   ♣ *Cantor's diagonal argument refers applies to the set of all infinite sequences of two characters. So consider the subset of those values who not only have no 0 or 9, but consist of only the digits 1 and 2. If this subset is uncountable then the larger set is as well.*

   To line them up nicely for the argument look at those subset of those who integer part is just 1, or 2. so numbers like

   $$2.2\overline{2} \textit{ or } 1.212212221222212222212222221222222\ldots$$

   This set is uncountable by Cantor's diagonal argument.

   Or you prefer, you can argue that the set is uncountable because, after the decimal point, it is encoded as an infinite number of independent non-trivial choices, (is the next digit 1*,* 2*,* 3*,* 4*,* 5*,* 6*,* 7*, or* 8*.)* ♣

8. Decide whether or not the set of all finite sequences of natural numbers is countable or uncountable.

   Show you claim is correct.

9. Decide whether or not the set of all infinite sequences of digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is countable or uncountable.

   Show you claim is correct.

10. Decide whether or not all strings of letters $\{a, b, c, d, \ldots, z\}$, like $ruddyfuddywuddyzoooop$, is countable or uncountable.

    (a string has finite length.)

    Show you claim is correct.

11. Just for Fun: Decide whether or not the set of all infinite non-decreasing sequences of digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is countable or uncountable.

    Example:

    $$1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9\ldots$$

    Show you claim is correct.

12. Just for Fun: Decide whether or not the set of all infinite sequences of integers $\{a_j\}_{j=0}^{\infty}$ such that $a_i \neq i$. is countable or uncountable.

    Show you claim is correct.