Exercises for Lectures 11 and 12

Relations of Functional Type.

We defined, one-to-one and onto, and, for finite sets, counted the number of each type.

Number of functions $A \to B$:

$$|B|^{|A|}$$

Number of one-to-one functions $A \to B$ *provided* $|B| \geq |A|$:

$$\frac{|B|!}{(|B| - |A|)!}$$

Number of one-to-one and onto functions $A \to B$ *provided* $|B| = |A|$:

$$|B|! = |A|!$$

Number of onto functions $A \to B$ *provided* $|B| \leq |A|$:

$$\sum_{k=0}^{|B|} \binom{|B|}{k} (-1)^k (|B| - k)^{|A|}$$

We showed that for two finite sets of equal cardinality, $|A| = |B|$, every one-to-one function is onto, and every onto-function is one-to-one.

For any set Q, and any function $f : Q \to \mathcal{P}(Q)$, we defined the set $R_f = \{q \in Q \mid q \notin f(q)\}$ and used R_f to show that there is no onto function from Q to $\mathcal{P}(Q)$. From this we concluded $|Q| < |\mathcal{P}(Q)|$. This led us to the the notion of countability vs. uncountability of infinite sets.

Exercises

1. Find an onto function $\mathbb{N} \to \mathbb{N}$ which is *not* the identity function.

2. Find an onto function $\mathbb{N} \to \mathbb{N}$ which is *not* one-to-one.

3. Find a one-to-one function function $\mathbb{N} \to \mathbb{N}$ which is *not* onto.

4. Find an onto function $f : \mathbb{N} \to \mathbb{N}$ which, for each $n \in \mathbb{N}$ we have $|f^{-1}(n)| = |\{m \in \mathbb{N} \mid f(m) = n\}| = 2$.

5. Definite a function $\mathcal{P}(\mathbb{N}) \to \mathbb{N}$ which is onto. What does this say about the cardinalities of $\mathcal{P}(\mathbb{N})$ and $\mathbb{N}$?

   ♣ *There are many ways to define this function. For instance, every subset A of $\mathbb{N}$ has a smallest element, min(A), so why not set $f(A) = min(A)$. Then the function is onto since for every n, $f(\{n\}) = \min(\{n\}) = n$.*

   For cardinalities, the existence of such an onto function says that $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$, which we know is true since $\mathcal{P}(\mathbb{N})$ is uncountable and $\mathbb{N}$ is merely countable. ♣

6. Define a function $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ which is one-to-one. What does this say about the cardinalities of $\mathcal{P}(\mathbb{N})$ and $\mathbb{N}$?

   ♣ *There are many ways to define this function, too. For instance,* $f(n) = \{n\}$. *This is one to one because each elements of the target is associated with at most one element - none if the cardinality of the set is not 1, and the value of the single element of the set if it has cardinality 1.*

   For cardinalities, the existence of such an one-to-one function says that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$, *the same inequality as in the previous, which we know is true since* $\mathcal{P}(\mathbb{N})$ *is uncountable and* $\mathbb{N}$ *is merely countable.* ♣

7. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and define a function $f : A \to \mathcal{P}(A)$ for which $R_f = A$. Show that $A \neq f(k)$ for all $1 \leq k \leq 10$.

8. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and define a function $f : A \to \mathcal{P}(A)$ for which $R_f = \emptyset$. Show that $\emptyset \neq f(k)$ for all $1 \leq k \leq 10$.