Discrete Mathematics D Term 2019

Lectures 07 and 08

Summary

We continued to talk about sets.

We introduced

$$\mathcal{P}_k A = \{X \subseteq A \mid |X| = k\}$$

the set of all k subsets of A and used a golden argument to show that it's cardinality is a binomial coefficient:

$$|\mathcal{P}_k A| = \frac{|A|!}{k!(|A|-k)!} = \binom{|A|}{k}$$

We discussed the arrangement of the binomial coefficients, most traditionally in *Pascal's Triangle*.

We showed some relations in Pascal's Triangle:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} &= 2^n \\
\sum_{k=0}^{n} (-1)^k \binom{n}{k} &= 0 \quad \text{another golden argument} \\
\sum_{k=0}^{n} \binom{n}{k} (j-1)^k &= j^n \quad n \text{ digit numbers in base } j \\
\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} &= (a+b)^n \quad \text{Binomial Theorem}
\end{aligned}$$

Also we discussed Russel's Paradox, which shows that even such a simple concept as "The set of all sets" runs into difficulties of well-definition.

We discussed as well the rumor that next Thursday's quiz will contain a proof with the double inclusion method.

Exercises for Lectures 07 and 08

In any these problems, it is sufficient to have an algebraic expression for the answer.

1. Write down the first 10 rows of Pascal's Triangle. Circle the following entries: $\binom{5}{3}$, $\binom{6}{2}$, $\binom{7}{4}$, $\binom{8}{4}$, and $\binom{10}{5}$,
2. List all the elements in $\mathcal{P}_2(\{a, b, c, d, e, f\})$
3. List all the elements in $\mathcal{P}_3(\{a, b, c, d, e, f\})$

4. List all the elements in $\mathcal{P}_3(\{a, b, c, d, e, f\})$
5. List all the elements in $\mathcal{P}_3(\mathcal{P}(\{a, b, c\}))$
6. What is the coefficient of x^3 in $(x + 1)^{15}$?
7. What is the coefficient of x^{15} in $(x^3 + 1)^{15}$?
8. What is the coefficient of x^{15} in $(x^5 + 1)^{15}$?
9. Six men are are at restaurant, a doctor, a lawyer, a bricklayer, a ditch-digger, a beggar, and a professor.

   Each gives a speech praising two of the professions represented at the table, but, courteously, not his own.

   How many ways are there to do this?

   ♣ *Each man must choose 2 of the other 5 professions, which he can do* $\binom{5}{2}$ *ways, and since there are six men, and the choices are independent, the ways this can be done is* $\binom{5}{2}^6$.

   Note: In an out-of-class discussion, a student argued that, in a speech, the order in which you praise the professions is important and ought to be counted. His argument was persuasive enough to mark his solution correct. What is his solution? ♣

10. How many 8 digit numbers in base 10 have exactly four ones, (like 01214161).
11. The are 120 passengers on the cruise ship Gigantic, which has three luxury lifeboats, the Nana, the Piñata, and the Santa Claus.

    a) How many ways would there be to arrange the passengers equally in the lifeboats. (40 per boat)

    ♣ *The Nana "chooses" 40 of the 120 passengers, so* $\binom{120}{40}$*, and for each way this is done, the Piñata "chooses" 40 of the remaining 80 passengers, so* $\binom{80}{40}$ *ways. And the Santa Claus has to take the remainder,* $\binom{40}{40} = 1$*, so there are* $\binom{120}{40}\binom{80}{40}$ *ways to arrange the passengers.* ♣

    b) The lifeboats are all smashed when the cruise ship hits an iceberg, so the passengers must be equally distributed instead onto three large scary indistinguishable rafts. How many ways are there do to this.

    ♣ *The difference here is that the rafts are indistinguishable, so the previous method would give over-counts for all the* 3! *ways of renaming the ships, so there are only*

    $$\frac{\binom{120}{40}\binom{80}{40}}{3!}$$

    ways of splitting the passengers onto three rafts. ♣

12. Circle each of the following which correctly completes the sentence. "Russel's Paradox ..."

    a) was a failure.

    b) does not impact the definition of the set of all natural numbers.

    c) does not impact the definition of the set of all subsets of natural numbers.

    d) does not impact the definition of finite sets.

    e) showed infinite sets do not exist.

13. How many eighteen digit numbers (base 10) have only three digits, used equally often, like $257,257,752,222,555,777$, or $000,111,222,222,222,000$.

    ♣ *Here is one way to do it:*

    We first choose the three digits, $\binom{10}{3}$ *ways. Next choose the positions of the smallest digit:* $\binom{18}{6}$ *ways. Next, from the remaining 12 positions, choose the positions of the next smallest digit:* $\binom{12}{6}$*. The the largest digits goes in the remaining 6 positions.*

    So $\binom{10}{3}\binom{18}{6}\binom{12}{6}$ *numbers.* ♣

14. A judge has 50 contestants and has to award a first place, a second place, and a third place. He asks in an online forum how many ways there are to do this.

    The first responder says that the only solution is to use the multiplicative principle with weak independence, choosing three prize winners in order: $50 \cdot 49 \cdot 48$.

    The second responder says that the first calculation is wrong and the first responder is an idiot and the number can be computed by using a binomial coefficient to choose the three winners, and then using the multiplicative principle to order three winners: $\binom{50}{3} \cdot 3!$.

    The third responder says the first two calculations are wrong, and that a correct way to do it is to count the ways to give all the contestants prizes up to 50'th place, and then to allow for the over-counts by dividing by the number of ways to award the unnecessary prizes. $\frac{50!}{47!}$.

    The fourth responder says that all the previous responders were wrong.

    Which one is correct?

    ♣ *All the mathematical solutions are correct, but the first responder says his method is the only one, so he is wrong there, and the second responder says the first responder is incorrect, which is incorrect even if the first responder is an idiot, the third responder says the first two methods are wrong, which is wrong. So the fourth responder is correct.* ♣

15. In the current cartoon on the webpage, relate (say by coloring) the regions in the diagram according to the fifth row of Pascal's Triangle.