Discrete Mathematics D Term 2019

Lectures 05 and 06

Summary

We introduced the cardinality of a set $|A|$, sets and showed

$$\begin{aligned}
|\mathcal{P}(X)| &= 2^{|X|} \\
|\mathcal{P}_k(X)| &= \binom{|X|}{k} = \frac{|X|!}{k!(|X|-k)!}, \\
|X \times Y| &= |X| \cdot |Y|, \quad \text{The Multiplicative Principle!} \\
|X^c| &= |U| - |X|, \\
|A|, |B| \leq |A \cup B| \leq |A| + |B| & \qquad 0 \leq |A \cap B| \leq |A|, |B|
\end{aligned}$$

We introduced the union, intersection and complement, and the four identities

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \qquad \overline{A \cup B} = \overline{A} \cap \overline{B}$$

We demonstrated the double inclusion method of showing two sets are equal and did several examples, including that

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B).$$

Set Theory

1. Let $Y = \{a, b, c, d, e\}$

   What is $|\mathcal{P}(\mathcal{P}(Y)) \cap Y|$?

   What is $|\mathcal{P}(\mathcal{P}(Y)) \cap \mathcal{P}(Y)|$?

   What is $\mathcal{P}_2(Y) \cup \mathcal{P}_3(Y)$?

   What is $|\mathcal{P}(\mathcal{P}(Y)) \times \mathcal{P}(Y)|$?

2. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. For each of the following, write down at least two elements in the set (if there are that many) and compute the cardinality

   (a) $X \times X$

   (b) $X \times X \times X \times X$

   (c) $\mathcal{P}(X)$

   (d) $\mathcal{P}(X) \times X$

   (e) $X \times \mathcal{P}_1(X) \times \mathcal{P}_1(\mathcal{P}_1(X)) \times \mathcal{P}_1(\mathcal{P}_1(\mathcal{P}_1(X)))$

(f) $X \times \mathcal{P}_2(X) \times \mathcal{P}_2(\mathcal{P}_1(X)) \times \mathcal{P}_2(\mathcal{P}_2(\mathcal{P}_2(X)))$

(g) $\mathcal{P}_0(X) \times \mathcal{P}_1(X) \times \mathcal{P}_2(X) \times \mathcal{P}_3(X)$

(h) $\mathcal{P}(X) \cup \mathcal{P}(\mathcal{P}(X))$ (careful)

(i) $\mathcal{P}(X) \cap \mathcal{P}(\mathcal{P}(X))$ (do you still have to be careful?)

(j) $\mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$

(k) $\mathcal{P}_{10}(\mathcal{P}_{11}(\mathcal{P}_{12}(X)))$

(l) $\mathcal{P}_{12}(\mathcal{P}_{11}(\mathcal{P}_{10}(X)))$

(m) $\mathcal{P}(\mathcal{P}(X) \times X)$

(n) $\mathcal{P}_2(\mathcal{P}_3(X) \times X)$

(o) $\mathcal{P}_3(\mathcal{P}_2(X) \times X)$ (Better stop before I run out of letters)

♣ Ok, here goes. . .

(a) $X \times X$
$(1, 1)$ and $(1, 2)$.
$|X \times X| = |X|^2 = 11^2$

(b) $X \times X \times X \times X$
$(1, 2, 3, 4)$ and $(4, 3, 2, 2)$.
$|X \times X \times X \times X| = |X|^4 = 11^4$

(c) $\mathcal{P}(X)$
$\{1, 2, 3\}$ and $\emptyset$.
$|\mathcal{P}(X)| = 2^{|X|} = 2^{11}$

(d) $\mathcal{P}(X) \times X$
$(\{1, 2, 3\}, 1)$ and $(\emptyset, 5)$.
$|\mathcal{P}(X) \times X| = 2^{|X|} \cdot 11 = 2^{11} \cdot 11$

(e) $X \times \mathcal{P}_1(X) \times \mathcal{P}_1(\mathcal{P}_1(X)) \times \mathcal{P}_1(\mathcal{P}_1(\mathcal{P}_1(X)))$
$(9, \{1\}, \{\{2\}\}, \{\{\{3\}\}\})$ and
$(8, \{1\}, \{\{2\}\}, \{\{\{3\}\}\})$.
$\binom{11}{1} = 11$, so 11^4.

(f) $X \times \mathcal{P}_2(X) \times \mathcal{P}_2(\mathcal{P}_1(X)) \times \mathcal{P}_2(\mathcal{P}_2(\mathcal{P}_2(X)))$
$(9, \{1, 2\}, \{\{2\}, \{3\}\}, \{\{\{1, 2\}, \{1, 3\}\}, \{\{1, 2\}, \{1, 4\}\}\})$ and
$(8, \{1, 2\}, \{\{2\}, \{3\}\}, \{\{\{1, 2\}, \{1, 3\}\}, \{\{1, 2\}, \{1, 4\}\}\})$.
$11 \cdot \binom{11}{2} \cdot \binom{\binom{11}{1}}{2} \cdot \binom{\binom{\binom{11}{2}}{2}}{2}$

(g) $\mathcal{P}_0(X) \times \mathcal{P}_1(X) \times \mathcal{P}_2(X) \times \mathcal{P}_3(X)$
$(\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\})$, and $(\emptyset, \{2\}, \{1, 2\}, \{1, 2, 3\})$.
$\binom{11}{0} \times \binom{11}{1} \times \binom{11}{2} \times \binom{11}{3}$.

(h) $\mathcal{P}(X) \cup \mathcal{P}(\mathcal{P}(X))$ (careful)
$\emptyset$, $\{1\}$.
$|\mathcal{P}(X) \cup \mathcal{P}(\mathcal{P}(X))| = |\mathcal{P}(X)| + |\mathcal{P}(\mathcal{P}(X))| - |\mathcal{P}(X) \cap \mathcal{P}(\mathcal{P}(X))| = |\mathcal{P}(X)| + |\mathcal{P}(\mathcal{P}(X))| - |\mathcal{P}(X \cap \mathcal{P}(X))| = |\mathcal{P}(X)| + |\mathcal{P}(\mathcal{P}(X))| - |\mathcal{P}(\emptyset)| = 2^{|X|} + 2^{2^{|X|}} - 2^0$

(i) $\mathcal{P}(X) \cap \mathcal{P}(\mathcal{P}(X))$ (do you still have to be careful?)
$\emptyset$, that's all there is.
1.
You *always* have to be careful!

(j) $\mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$
$\emptyset, \{\{\{1\}\}\}$.
$2^{(2^{(2^{11})})}$ (is very very big.)

(k) $\mathcal{P}_{10}(\mathcal{P}_{11}(\mathcal{P}_{12}(X)))$
Nothing here folks. (X has no 12 subsets.)

(l) $\mathcal{P}_{12}(\mathcal{P}_{11}(\mathcal{P}_{10}(X)))$
X has 11 10-subsets, so the $\mathcal{P}_{11}(\mathcal{P}_{10}(X)$ has just one element, (all eleven of them) so there are no 12-subsets of that. So nothing here either.
If you don't like that, do $\binom{\binom{\binom{11}{10}}{11}}{12} = \binom{\binom{11}{11}}{12} = \binom{1}{12} = 0$

(m) $\mathcal{P}(\mathcal{P}(X) \times X)$
$\{(\{1\}, 1)\}$, and $\{(\{1\}, 1), (\{1, 5\}, 2)\}$.
$2^{2^{11} \cdot 11}$.

(n) $\mathcal{P}_2(\mathcal{P}_3(X) \times X)$
$\{(\{1, 2, 3\}, 3), (\{1, 2, 3\}, 4)\}$ and $\{(\{1, 2, 3\}, 3), (\{3, 4, 5\}, 4)\}$.
$\binom{\binom{11}{3} \cdot 11}{2}$.

(o) $\mathcal{P}_3(\mathcal{P}_2(X) \times X)$ Now, that wasn't so bad...
$\{(\{1, 2\}, 3), (\{2, 3\}, 4), (\{2, 3\}, 5)\}$ and $\{(\{1, 2\}, 3), (\{2, 3\}, 4), (\{2, 3\}, 7)\}$.
$\binom{\binom{11}{2} \cdot 11}{3}$. ♣

3. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 3, 6, 8\}$. Compute $A \cup B$, $A \cap B$, $\mathcal{P}(A \cap B)$, $\mathcal{P}(A) \cap \mathcal{P}(B)$

4. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 9, 10\}$ be the universe from which all the elements are drawn. Let $A = \{5\}$ and $B = \{3, 4, 5, 6\}$.

   Compute $A \cup B$, $A \cap B$, $A \cap B^c$, $A^c \cap B^c$, and $A \cap (B \cap A^c)$.

5. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 9, 10\}$ be the universe from which all the elements are drawn. Let $A = \{5\}$ and $B = \{3, 4, 5, 6\}$. Write down three different expressions involving A, B, $\cup$, $\cap$, and $(-)^c$ which denote the empty set.

6. Let A and B be sets. Prove that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ using the double inclusion method.

7. Let A and B be sets. Prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ using the double inclusion method.

   ♣ *First we show* $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$*, so let* $x \in (A \cup B) \cap C$*. Thus* $x \in A \cup B$ *and* $x \in C$*. Since* $x \in A \cup B$*, we have* $x \in A$ *or* $x \in B$*.*

Case 1: Let $x \in A$. Then, since $x \in C$, we have $x \in A \cap C$, hence $x \in (A \cap C) \cup (B \cap C)$.

Case 2: Let $x \in B$. Then, since $x \in C$, we have $x \in B \cap C$, hence $x \in (A \cap C) \cup (B \cap C)$ in this case as well.

So $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$.

Second we show $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$, so let $y \in (A \cap C) \cup (B \cap C)$, so there are two cases.

Case 1: $y \in A \cap C$, so $y \in A$ and $y \in C$. Since $y \in A$, we have $y \in A \cup B$, so $y \in (A \cup B) \cap C$.

Case 2: $y \in B \cap C$, so $y \in B$ and $y \in C$. Since $y \in B$, we have $y \in A \cup B$, so $y \in (A \cup B) \cap C$ in this case as well.

So $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$.

Therefore, by the double inclusion method, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
♣

8. Let A and B be sets. Prove that $(A \cap B)^c = A^c \cup B^c$ using the double inclusion method.

9. Let A and B be sets. Prove that $(A \cup B)^c = A^c \cap B^c$ using the double inclusion method.

10. Let $P = \{n \in \mathbb{N} \mid n = 6i, i \in \mathbb{N}\}$, $Q = \{m \in \mathbb{N} \mid m = 3j, j \in \mathbb{N}\}$, and $R = \{l \in \mathbb{N} \mid l = 2k, k \in \mathbb{N}\}$.

    Show with the double inclusion method that $P = Q \cap R$.

    [Hint: Somewhere in the argument, you might want to use that the product of two odd numbers must be odd.]

11. In class we proved $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

    Try to prove $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

    (If you don't think if is true, try to find two sets A and B for which it is false.)

    ♣ *It is false. Try $A = \{1\}$ and $B = \{2\}$, then $\mathcal{P}(A \cup B)$ contains $\{1, 2\}$ as an element, but neither of $\mathcal{P}(A)$ or $\mathcal{P}(B)$ does.* ♣