Discrete Mathematics D Term 2019

Lectures 03, 04

Summary

We discussed other aspects the multiplicative principle.

- weak independence
- method with controlled over-counting (e.g. anagrams with repeated letters)
- method with controlled illegal elements

We also discussed the relationship between the magic trick and the binary numbers. In particular

- two methods of Base conversion
- arithmetic in other bases.

We introduced sets, their notation, the idea of being well-defined, subsets, and the power set. We showed by the multiplicative principle that for a finite set A with n elements that its power set $\mathcal{P}(A)$ has 2^n elements. We took special care of the empty set.

Exercises on Lectures 01 and 02

Exercises on Multiplicative principle

1. In each of the following, decide whether the sequence of choices is independent, weakly independent, or neither.

   (a) Choose a number from 1 to 10,000, then a number within 10 of your first choice, then a number within 10 of your second choice, (you can choose the same number more than once.)

   (b) 25 ballerina's are holding hands in a circle. Choose a dancer, then one of the dancers she is holding hands with, then any dancer neither of them are holding hands with.

   (c) 25 ballerina's are holding hands in a circle. Choose any pair of ballerinas, holding hands or not, then choose a pair of ballerinas who are holding hands.

   (d) 25 ballerina's are holding hands in a circle. Choose any pair of ballerinas, holding hands or not, then choose a pair of ballerinas who are holding hands, neither of which were chosen before.

2. How many anagrams of 'placebo' are there?

3. How many anagrams of ‘placebo’ have all the vowels together?
4. How many anagrams of ‘placebo’ have at least two syllables (at least one pair of vowels separated by a consonant.)
5. How many anagrams of ‘Mississippi’ are there?
6. How many anagrams of ‘Mississippi’ are there which have no double p’s?
7. How many anagrams of ‘paradimethylaminobenzaldehyde’ are there?

$$HappySt.Patrick'sDay!$$

Exercises on Number Bases

1. Add 101010101 to 11101110111 in binary. Check your answer in decimal.
2. Subtract 101010101 from 11101110111 in binary. Check your answer in decimal.
3. Multiply 1010111101 by 1011 in binary. Check your answer in decimal.
4. Convert 1215, 1492, 1688, and 1861 to binary by both methods.
5. Convert 1776 to bases 3, 8 and 16. For base 16 use for the “digits” 10, 11, 12, 13, 14, 15 and the symbols A, B, C, D, E and F.)

   ♣ *For converting to base 3, repeatedly divide by* 3 *and consider the sequence of remainders:*

$$\begin{aligned}
1776 &= 3 \cdot 592 + 0 \\
592 &= 3 \cdot 197 + 1 \\
197 &= 3 \cdot 65 + 2 \\
65 &= 3 \cdot 21 + 2 \\
21 &= 3 \cdot 7 + 0 \\
7 &= 3 \cdot 2 + 1 \\
2 &= 3 \cdot 0 + 2
\end{aligned}$$

   So it is $2,111,220$ *in base 3.*

For 8 and 16 why not convert to base two first:

$$\begin{aligned}
1776 &= 2\cdot 888+0\\
888 &= 2\cdot 444+0\\
444 &= 2\cdot 222+0\\
222 &= 2\cdot 111+0\\
111 &= 2\cdot 55+1\\
55 &= 2\cdot 27+1\\
27 &= 2\cdot 13+1\\
13 &= 2\cdot 6+1\\
6 &= 2\cdot 3+0\\
3 &= 2\cdot 1+1\\
1 &= 2\cdot 0+1
\end{aligned}$$

So it is 11011110000 *in base 2. So base 8 it is* $11,011,110,000$ *with the 'digits' being the triples in base 2, so* $3,360$. *And base 16 is* $110,1111,0000$ *where the digits are the quadruples in base 2, so* $6F0$. ♣

6. (Just for fun...) In the early days of computers, a system was proposed to base the architecture on memory locations having three states, $+$, $-$ and 0, "pos", "neg" and "nul", for positive, negative and neutral. This would inspire a number system with base 3 and in which the "six digit number" $+--+0+$ would be $3^5-3^4-3^3+3^2+3^0 = 243-81-27+9+1 = 253-107 = 146$.

   (a) Write out the numbers from 0 to 27 in this system.

   (b) Find 11, 22 and 33 in this trinary system.

   (c) Add your answers for 11 and 22 together and see if you get the correct answer.

   (d) Can you figure out how to multiply your answers for 11 and 22 to get $11\cdot 22 = 242 = 243-1 = +0000-$?

♣ *We have to count up to* 27, *that is pos nul nul nul: (across then down.)*

$$\begin{array}{rrr}
 & 0 & + \\
+- & +0 & ++ \\
+-- & +-0 & +-+ \\
+0- & +00 & +0+ \\
++- & ++0 & +++ \\
+--- & +--0 & +--+ \\
+-0- & +-00 & +-0+ \\
+-+- & +-+0 & +-++ \\
+0-- & +0-0 & +0-+ \\
+00- & +000 &
\end{array}$$

for b) we already have 11 as $++-$, *or* $9+3-1$, *and 22 as* $+-++$ *or* $27-9+3+1$. *We can add them like this using* $+\oplus- = 0$, *and* $+\oplus+ = +-$, *etc.*

$$\begin{array}{rcccc} & & + & + & - \\ \oplus & + & - & + & + \\ \hline & + & + & - & 0 \end{array} \text{(don't forget the carries)}$$

Now we convert 33 *to see if it is correct. Here we again do repeated division, but instead of remainders, we look for the nearest multiple of* 3*:*

$$\begin{aligned} 33 &= 3\cdot 11+0 \\ 11 &= 3\cdot 4-1 \\ 4 &= 3\cdot 1+1 \\ 1 &= 3\cdot 0+1 \end{aligned}$$

So 33 is indeed $++-0$.

For d)

$$\begin{array}{cccccc} & & & + & + & - \\ \otimes & & + & - & + & + \\ \hline & & & + & + & - \\ & & + & + & - & \\ & - & - & + & & \\ + & + & - & & & \\ \hline + & 0 & 0 & 0 & 0 & - \end{array} \quad \text{(don't forget the carries and } +\oplus+\oplus+ = +0\text{)}$$

And you know what, that was fun. ♣

7. (Just for fun...) Old Professor Fate is so indecisive, he cannot decide between 2 and 3 for a number base. So he ends up with a system in which he uses digits 0 and 1 in the even positions, but allows 0, 1 and 2 in odd positions.

   (a) Count from 1 to 25 in the Professor Fate numbers.

   (b) For the number 110011000 in Professor Fate's system, what is the value associated with each of the 1's.

   (c) Convert 110011000 to decimal numbers.

   (d) Convert 2018 to Professor Fate numbers. Try to adapt both methods we learned for base conversion to this system.

   (e) What single base are the Professor Fate numbers most related to, and why?

Exercises on Set Theory

1. Discuss whether or not each of these sets is well defined.
   - The set of shoes shoes belonging to WPI professors.
   - The set of hairs on your head.
   - The set of cells in your body.
   - The set of trees on campus.
   - WPI buildings.

2. Discuss whether or not each of these sets is well defined.
   - The set of ages of the members of this class.
   - The set of numbers larger than six.
   - The set of prime numbers which are divisible by six.
   - The set of rational numbers which are complex.
   - The set of all numbers which have been been interesting to some human being.

3. Let $X = \{0, 1, 2\}$

   Write down all elements of $\mathcal{P}(X)$.

4. Let $Z = \{0, 1\}$.

   Write down all elements of $\mathcal{P}(\mathcal{P}(Z))$.

5. Write down all elements of $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))))$. (There are 16 of them)

6. Let $Y = \{a, b, c, d, e\}$

   Write down all elements of $A \in \mathcal{P}(Y)$ with 3 elements.

   How many elements does $\mathcal{P}(Y)$ have?

   How many elements does $\mathcal{P}(\mathcal{P}(Y))$ have?

7. Let $\mathbb{N}$ be the natural numbers, (starting with 0) and let $\emptyset$ denote the empty set. For each of the following indicate if the statement is true or false.

   ___ $\emptyset \in \mathbb{N}$

   ___ $\emptyset \in \mathcal{P}(\mathbb{N})$

   ___ $\emptyset \subseteq \mathcal{P}(\mathbb{N})$

   ___ $\mathcal{P}(\emptyset) \subseteq \mathcal{P}(\mathbb{N})$

   ___ $\mathcal{P}(\emptyset) \in \mathcal{P}(\mathcal{P}(\mathbb{N}))$

   ♣ *For these, it really doesn't matter which set is referred to, as long as is it non-empty.*

   First is FALSE. $\emptyset$ is not a number, so it is not in the set of natural numbers.

Second is TRUE. $\emptyset \subseteq \mathbb{N}$ *so* $\emptyset \in \mathcal{P}(\mathbb{N})$.

Third is TRUE. $\emptyset$ *is a subset of every set.*

For the Fourth, you must check that every element of $\mathcal{P}(\emptyset)$ *is also an element of* $\mathcal{P}(\mathbb{N})$. *The only element of is* $\emptyset$, *that is,* $\mathcal{P}(\emptyset) = \{0\}$. *And* $\emptyset \subseteq \mathbb{N}$, *so* $\emptyset \in \mathcal{P}(\mathbb{N})$.
So Fourth is TRUE.

(Just because it is True/False, doesn't mean you can decide right away without thinking it through.)

For the fifth, it is like pealing an onion: we want to know if $\{\emptyset\} \subseteq \mathcal{P}(\mathbb{N})$, *which is whether* $\emptyset \in \mathcal{P}(\mathbb{N})$, *which is when* $\emptyset \subseteq \mathbb{N}$, *which is of course TRUE. The empty set is a subset of every set.* ♣