Exercises for Lectures 17 and 18

These lectures considered functions and cardinality.

$|A| < |B|$: There is a one-to-one function from A to B.

or

There is an onto function from B to A.

$|A| = |B|$: $|A| \leq |B|$ and $|B| \leq |A|$

or

There is a one-to-one onto function from A to B.

For any set A, and any function $f : A \to \mathcal{P}(A)$, we defined the set $B_f = \{a \in A \mid a \notin f(a)\}$ and used B_f to show that there is no onto function from A to $\mathcal{P}(A)$.

We described Cantor's diagonal argument (a variation on this) to show that $\mathbb{R}$ is uncountable.

We showed that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all countably infinite.

We showed that $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$ are both uncountable.

We showed that there are infinitely many different uncountable cardinals.

We mentioned that Peano showed (surprise!) that the set of points on a line and the set of points in the plane are of the same cardinality.

We showed that if A and B are countably infinite, then $A \cup B$ and $A \times B$ are both countably infinite.

We showed that if you have countably infinite collection of countably infinite sets, A_1, A_2, A_3, A_4, A_5, ..., then the set

$$\bigcup_{i=1}^{\infty} A_i$$

is countably infinite.

But noted that then the set

$$A_1 \times A_2 \times A_3 \times \cdots$$

is uncountable, uncountable even if the sets A_i are all of cardinality 2!

Exercises

1. Definite a function $\mathcal{P}(\mathbb{N}) \to \mathbb{N}$ which is onto. What does this say about the cardinalities of $\mathcal{P}(\mathbb{N})$ and $\mathbb{N}$?

2. Definite a function $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ which is one-to-one. What does this say about the cardinalities of $\mathcal{P}(\mathbb{N})$ and $\mathbb{N}$?

3. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and define a function $f : A \to \mathcal{P}(A)$ for which $B_f = A$. Show that $A \neq f(k)$ for all $1 \leq k \leq 10$.

4. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and define a function $f : A \to \mathcal{P}(A)$ for which $B_f = \emptyset$. Show that $\emptyset \neq f(k)$ for all $1 \leq k \leq 10$.

5. Let $X = \{n \in \mathbb{N} \mid n \text{ is evenly divisible by 1 million}\}$. Define a function which shows that X is countably infinite.

   Is this compatible with the fact that $X \subseteq \mathbb{N}$?

6. Let $Y = \{n \in \mathbb{N} \mid \text{the decimal representation of } n \text{ contains no 7.}\}$. Define a function which shows that X is countably infinite.

7. Let $Z = \{n \in \mathbb{R} \mid \text{the decimal representation of } n \text{ contains only 4's and 7's.}\}$. Define a function which shows that X is uncountable.

8. Find a one-to-one onto function with domain $A \times (B \times C)$ and target $(A \times B) \times C$.

9. Label each of the following as finite, countable, or uncountable.

   (a) The set of subsets of $\mathbb{Q}$.

   (b) $\mathcal{P}(\mathbb{R})$.

   (c) The set of functions with domain $\mathbb{Z} \times \mathbb{Z}$ and target $\{0,1\} \times \{0,1\}$.

   (d) The set of functions with domain $\{0,1\} \times \{0,1\}$ and target $\mathbb{Z} \times \mathbb{Z}$.

   (e) the set of positive prime integers.

   (f) the set of finite subsets of $\mathbb{Z}$. [Hint: Write it as a countable union of finite sets.]

   (g) The set of non-increasing infinite sequences of elements of $\mathbb{N}$, that is sequences
   $$\{a_k\}_{k \in \mathbb{N}}$$
   such that $a_{k+1} \leq a_k$ for all k.

   (h) The set of non-decreasing infinite sequences of elements of $\mathbb{N}$, that is sequences
   $$\{a_k\}_{k \in \mathbb{N}}$$
   such that $a_{k+1} \geq a_k$ for all k.

10. Let $A_i = \{1, 2, 3, \ldots, i\}$. Which of the following are countably infinite.

   (a) $\displaystyle\bigcup_{i=1}^{\infty} A_i$

   (b) $\displaystyle\bigcap_{i=1}^{\infty} A_i$

   (c) $\times_{i=1}^{\infty} A_i$

   (d) $\displaystyle\bigcup_{i=1}^{\infty} \mathcal{P}(A_i)$

   (e) $\displaystyle\bigcup_{i=1}^{\infty} \mathcal{P}^i(A_i)$

(f) $\mathcal{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$

11. **Challenging: Don't do this unless you found the problems above were too easy.**

    Let

    $$\{X \in \mathcal{P}(\mathbb{R}) \mid \text{X is the set of real roots of some polynomial with rational coeficients.}\}$$

    Find three elements of X, each with different cardinalities.

    Show that X is countably infinite.