Summary for Lectures 15 and 16

We continued discussing functional relations, specifically counting functions from finite sets of various types.

We called this collection of problems the "Santa Claus Problems". If you look it up in the literature, you might find it under "The Twelve-fold Way".

We have a few types to go, but to make the summary easier I'll include all of them here, as well as their exercises:

Counting the number of functions $f : T \to C$.

	Ordinary	One-to-one	Onto
Normal	$\lvert C\rvert^{\lvert T\rvert}$	$\frac{\lvert C\rvert!}{(\lvert C\rvert-\lvert T\rvert)!}$	$\sum_{j=0}^{\lvert C\rvert}(-1)^j\binom{\lvert C\rvert}{j}(\lvert C\rvert-j)^{\lvert T\rvert}=(\lvert C\rvert!)\left\{{\lvert T\rvert \atop \lvert C\rvert}\right\}$
Domain Unlabeled $\lvert T\rvert = d$	$\binom{d+\lvert C\rvert-1}{\lvert C\rvert-1}$	$\binom{\lvert C\rvert}{d}$	$\binom{d-1}{\lvert C\rvert-1}$
Target Unlabeled $\lvert C\rvert = k$	$\sum_{j=1}^{k}\left\{{\lvert T\rvert \atop j}\right\}$	1 or 0	$\left\{{\lvert T\rvert \atop k}\right\}$
Both Unlabeled	$p_k(d+k)$	1 or 0	$p_k(d)$

In this table $\left\{{n \atop k}\right\}$ is a *Stirling Number of the second kind*, also called a *set partition number* and $p_k(n)$ is an *integer partition number*.

We saw they satisfy recursions similar to that for the binomial coefficients:

$$\left\{{n \atop k}\right\} = \left\{{n-1 \atop k-1}\right\} + k\left\{{n-1 \atop k}\right\} \qquad p_n(k) = p_{n-1}(k-1) + p_{n-k}(k)$$

$\left\{{\lvert T\rvert \atop k}\right\}$:

1
1 1
1 3 1
1 7 6 1
1 15 25 10 1
1 31 90 65 15 1
1 63 301 350 140 21 1
1 127 966 1701 1050 266 28 1

$$
p_d(k): \quad
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 2 & & 1 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 2 & & 1 & & 1 & & \\
 & 1 & & 3 & & 4 & & 3 & & 2 & & 1 & & 1 & \\
1 & & 4 & & 5 & & 5 & & 3 & & 2 & & 1 & & 1
\end{array}
$$

Exercises

1. Find the next row in the table for the Stirling numbers of the second kind above, $\left\{ {9 \atop k} \right\}$.

2. Find the next row in the table for number of integer partitions, the integer partitions of 9 size k; $p_k(9)$.

3. Consider the functions from with domain $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and target $Y = \{a, b, c, d, e, f\}$. Fill in the table of the 12-fold way for this case.

	Ordinary	One-to-one	Onto
Normal			
Domain Unlabeled			
Target Unlabeled			
Both Unlabeled			

4. Consider the functions from with domain $Y = \{a, b, c, d, e, f\}$ and target $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Fill in the table of the 12-fold way for this case.

	Ordinary	One-to-one	Onto
Normal			
Domain Unlabeled			
Target Unlabeled			
Both Unlabeled			

5. In an experiment there are 25 tulip bulbs, *tulipa agenensis*, which are to be treated with either one of 7 pesticides, or no pesticide. How many ways are there to perform the experiment?

6. A gardener has 25 tulip bulbs of different species, which he treats with from 0 to 7 doses of a pesticide. How many ways are there to for him to treat the plants?

   What would be the interpretation if the relevant function is onto?

7. A Florist makes an arrangement of 25 tulips which are to be decorated with 8 red ribbons, each ribbon to be attached to some tulip. How many types of ribbon arrangements are there?

8. A botanist has custody of 25 rare tulip specimens. He has 65 fertilize pellets. How many ways can he fertilize the tulips if he uses all the pellets and each tulip receives at least two pellets.

9. How many ways are there to distribute 8 tasks among 4 identical robots so that each task is to be completed by a single robot, and every robot is given at least one task.

10. How many ways are there to distribute 5 tasks among 3 identical robots so that each task is to be completed by a single robot.

11. How many ways are there to distribute 8 identical tasks among 4 identical robots so that each task is to be completed by a single robot, and every robot is given at least one task.

12. How many ways are there to distribute 5 identical tasks among 3 identical robots so that each task is to be completed by a single robot.

For the following problems express your answer in terms of $\left\{ n \atop k \right\}$, $p_k(d)$, *or binomial coefficients as appropriate.*

13. How many ways are there to distribute 35 quarters into 6 teacups? What about 16 tea cups?

14. At a carnival game, quarters are tossed into dishes. For each quarter landing in the red dish the tosser wins a piece of candy, for each landing in the blue dish, the tosser receives a dollar, for each landing in the yellow dish, the tosser wins $200, and for those ending up on the floor, the tosser receives nothing. If 65 quarters are tossed, how many outcomes are there?

15. Candy is distributed a kindergarten so that the children get 5 pieces on average. If the children are distinguishable, how many ways are there to do this.

    What if the children are indistinguishable?

    What if the children are distinguishable but each child must get at least two pieces of candy?

16. Twelve people attend a party. While at the party some guests photograph some of the other guests and post the pictures online at their own site.

    How many ways are there to do this? [Hint: There are quite a few.]

17. Show that $\left\{ n \atop 2 \right\} = 2^{n-1} - 1$.

18. Show that $\left\{ n \atop n-1 \right\} = \binom{n}{2}$.

19. Show that $\left\{ n \atop n-2 \right\} = \binom{n}{3} + \binom{n}{4}\left\{ 4 \atop 2 \right\}$.

20. Show that find a formula for $\left\{ n \atop n-3 \right\}$ similar that of the previous exercise.