

Summary for Lectures 13 and 14

These lectures concerned structures on sets.

We considered *Relations*, $R \subseteq A \times B$.

Every relation can be indicated by a diagram with the elements of A and B drawn as nodes, with an arrow from a to b , if $(a, b) \in R$.

We drew the *state diagram* for the relation on the states of the Wolf/Goat/Cabbage problem, and showed how this aided in analyzing the number and types of solutions.

We described some important qualities relations on $A \times A$ may have:

- Reflexive; $a \in A \Rightarrow (a, a) \in R$. (diagram has loops.)
- Antireflexive; $a \in A \Rightarrow (a, a) \notin R$. (diagram has loops.)
- Symmetric; $(a, b) \in R \Rightarrow (b, a) \in R$. (Diagram is a graph.)
- Antisymmetric; $(a, b) \in R \Rightarrow (b, a) \notin R$. (Diagram is a graph.)
- Transitive; $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$. (Diagram becomes rather hard to draw.)

We gave examples of relations which were reflexive, transitive and had the property that $(a, b) \in R, (b, a) \in R \Rightarrow a = b$, which made A a *partially ordered set*, (also called a poset by English manglers.)

We also gave examples of *equivalence relations*, which are reflexive, symmetric and transitive, and showed how every set with an equivalence relation is partitioned into equivalence classes.

Exercises for Lectures 13 and 14

1. Consider the Wolf/Goat/Cabbage problem as we did in class, but in this version, the viking's short term memory is just powerful enough that he can remember the move he just completed, and he will absolutely refuse to undo do. So if he has just rowed the goat across the river, the next move must not involve the goat. Describe the states of this new problem and draw the state diagram.
2. Let $A = \{1, 2\}$ and $B = \{1, 2, 3, 4\}$ and definite a relation $R \subseteq \mathcal{P}(A) \times \mathcal{P}(B)$ by setting $(X, Y) \in R$ if $X \subseteq Y$. Draw the diagram of the relation.
3. Let $A = \{1, 2, 3, 4, 5\}$
 - (a) How many relations on $A \times A$ are there?
 - (b) Find a relation which is a partial order.
 - (c) Find a relation which is reflexive and symmetric but not transitive.
 - (d) Find a relation which is symmetric and transitive but not reflexive.

- (e) Find a relation which is transitive and reflexive but not symmetric.
- (f) Find a relation which is antisymmetric and transitive.
4. Define a relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ by saying $(n, m) \in R$ if the (decimal) number $n - m$ ends in a zero. Show that R is an equivalence relation and describe the equivalence classes.
 5. Define a relation $S \subseteq \mathbb{Z} \times \mathbb{Z}$ by saying $(n, m) \in R$ if the (binary) number $n - m$ ends in a zero. Show that R is an equivalence relation and describe the equivalence classes.
 6. Define a relation $T \subseteq \mathbb{Z} \times \mathbb{Z}$ by saying $(n, m) \in R$ if the (decimal) number $n + m$ ends in a zero. Is T is an equivalence relation? Why or why not.
 7. Define a relation $U \subseteq \mathbb{Z} \times \mathbb{Z}$ by saying $(n, m) \in R$ if the (decimal) number $n - m$ ends in a 1. Is U is an equivalence relation? Why or why not.
 8. Don't do this problem: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many equivalence relations $R \subseteq A \times A$ are there?
 9. We showed in class that if L is a set of statements, that $p \Rightarrow q$ defines a partial order on $L \times L$. Does $\neg(p \rightarrow q)$ also define a partial order? What about $q \Rightarrow p$. What about $p \vee q$?