Lectures 10 and 11

We continues to discuss logical implication: $p \Rightarrow q$, IF p THEN q, or p IMPLIES q.

$$(p \Rightarrow q) = (q \vee \neg p)$$

We discussed the cannonball stacking problem.

We showed how it was reasonable to suspect that the number of cannonballs in a square based stack of height n is given by $\dfrac{n(n+1)(2n+1)}{6}$.

We proved this statement by *Mathematical Induction*.

We proved $n^3 - n$ is evenly divisible by 6 by for all n by induction. We also gave a very convincing non-inductive proof.

As an important example we proved the Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Exercises for Lectures 10 and 11

1. Write $p \Rightarrow (p \wedge r)$ as an expression using only $\vee$, $\wedge$ and $\neg$.
2. Write $p \vee (q \wedge r) \Rightarrow (p \wedge \neg q))$ as an expression using only $\vee$, $\wedge$ and $\neg$.
3. Consider the statements $(p \Rightarrow q) \Rightarrow r$ and $p \Rightarrow (q \Rightarrow r)$. Does one imply the other? Are they equivalent?
4. Let p_n be the statement $2+4+6+8+\cdots+(2n) = n(n+1)$. What is p_1? What is p_2. What is p_{10}? Which of them are true?
5. Let p_n be the statement $2+4+6+8+\cdots(2n) = n(n+1)$. Show that p_n is true for all $n \geq 1$ by induction.
6. Let n be a natural number and let p_n be the statement "The number $n^3 - n$ is evenly divisible by 3".

   What is statement p_2?

   What is statement p_5?

   Are they both true?
7. Let n be a natural number and let p_n be the statement "The number $n^3 + 2n$" is evenly divisible by 3.

   Show that p_n is true for all n by induction.
8. Let p_n be the statement "In a party with n people, if every guest shakes hands once with every other guest, there must be $n(n-1)/2$ handshakes during the party."

   What is statement p_5? It is true?

   Show all statements p_n are true by induction for $n > 0$.

9. Prove by induction that $2n^3 + 3n^2 + n$ is always evenly divisible by 6.

10. Show by induction that $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$

11. Consider the statement p_n defined by

$$1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)(n-2)(n-3)}{24} = 2^n$$

Check the truth of p_n for $n = 0$, $n = 1$, $n = 2$, $n = 3$, and $n = 4$.

What can you conclude from this about the truth of p_n for all n?

12. Consider a collection of n straight lines drawn haphazardly in the plane, so that no two are parallel and no three or more intersect in a single point. How many points of intersection does this collection of n lines have? Note, the intersection point still exists even if it is off the paper.

Consider small cases, and look for a formula. Prove your formula using induction.

13. Use induction on m to show that the number of functions from n-set to an m-set is m^n for $m \geq 1$.

14. Use induction on n to show that

$$p \vee (q_1 \wedge q_2 \wedge \cdots \wedge q_n) = (p \vee q_1) \wedge (p \vee q_2) \wedge \cdots \wedge (p \vee q_n).$$

15. Use induction on n to show that

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$$

16. Use induction on n to show that

$$\neg(p_1 \wedge p_2 \cap \cdots \wedge p_n) = (\neg p_1) \vee (\neg p_2) \vee \cdots \vee (\neg p_n).$$