

Lectures 23 and 24

Continuing on Recursion.

We continued to discuss the Towers of Hanoi – three spindle version.

We counted the number of states in optimal play by considering the recursive dependence:

For disk k to move from spindle a to spindle b in optimal play, ($T_k \in C_k = \{a, b\}$) disk $k - 1$ will be either on top of disk k , that is on spindle T_k , or the third spindle, $T_k \in \overline{C_k}$. so $T_{k-1} \in C_{k-1} = \overline{C_k} \cup \{T_k\}$.

This dependence satisfies the weak version of the multiplicative principle, assigning spindles from largest to smallest there are always two choices for the next spindle, so there are 2^n states possible in optimal play.

We can use $C_{k-1} = \overline{C_k} \cup \{T_k\}$ to test whether states occur in optimal play.

We also showed how to move optimally toward a specific state by assigning goals largest to smallest, and full filling them smallest to largest. So if the goal is $(1, 1, 1, 1)$ and the current state is $(0, 2, 0, 0)$, Then assigning goals as subscripts we have $(0_1, 2, 0_2, 0_1)$ and optimal play proceeds, with a progress recorded by a binary vector with $b_n = 1$ if the goal is achieved (no subscript), $b_k = 0$ if not, (subscript).

0_1	2	0_2	0_1	0	1	0	0
0_1	2	0_2	1	0	1	0	1
0_1	2	2	1_2	0	1	1	0
0_1	2	2	2	0	1	1	1
1	2_1	2_0	2_1	1	0	0	0
1	2_1	2_0	1	1	0	0	1
1	2_1	0	1_0	1	0	1	0
1	2_1	0	0	1	0	1	1
1	1	0_1	0_2	1	1	0	0
1	1	0_1	2	1	1	0	1
1	1	1	2_1	1	1	1	0
1	1	1	1	1	1	1	1

We noted that the goal vector incremented identically to incrementing binary numbers, so the binary subtraction $1111 - 0100 = 1011$ (decimal 11) gives the number of moves required.

We also solved the reverse problem: e.g. after 20 moves of optimal play, from $(0, 0, 0, 0, 0)$ to $(1, 1, 1, 1, 1)$, what is the state, T ?

We used the recursions:

$$C_{k-1} = \overline{C_k} \cup \{T_k\} \quad b_k = b_{k-1} \iff T_k = T_{k-1}$$

and the binary representation of 20, 10100.

b	1	0	1	0	0
C	$\{0, 1\}$	$\{1, 2\}$	$\{2, 0\}$	$\{1, 0\}$	$\{1, 2\}$
T	1	2	0	1	1

to recursively generate the entries C_k and T_k .

Lastly we showed how to draw the state diagram for the towers of Hanoi problem.

Exercises for Lectures 23 and 24

1. In the three spindle towers of Hanoi with four disks, draw the states encoded by the vectors $T = (0, 0, 0, 1)$, $T = (0, 1, 2, 0)$ and $T = (0, 1, 0, 2)$.
2. What moves optimally take state $(1, 0, 0, 1)$ to state $(1, 1, 1, 1)$?
3. From state $(1, 0, 1, 1, 0, 0, 2, 0, 2)$, to move optimally to $(1, 1, 1, 1, 1, 1, 1, 1, 1)$, what are the next two moves?
4. Does state $(1, 0, 2, 2, 0)$ occur in optimal play from $(0, 0, 0, 0, 0)$ to $(1, 1, 1, 1, 1)$?
5. How many moves does it take to move from state $(1, 0, 2, 1, 2)$ to $(1, 1, 1, 1, 1)$?
6. How many moves does it take to move from state $(1, 0, 2, 1, 2)$ to $(2, 1, 2, 0, 1)$?