Lectures 11 and 12

We discussed logical implication: $p \Rightarrow q$, IF p THEN q, or p IMPLIES q.

$$(p \Rightarrow q) = (q \vee \neg p)$$

We showed the double implication method for showing $p \Leftrightarrow q$ by showing separately that $p \Rightarrow q$ and $q \Rightarrow p$

We discussed the cannonball stacking problem.

We showed how it was reasonable to suspect that the number of cannonballs in a square based stack of height n is given by $\dfrac{n(n+1)(2n+1)}{6}$.

We proved this statement by *Mathematical Induction*.

Exercises for Lectures 9 and 10

1. Use the double implication method to show that
$$p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r).$$

2. Use the double implication method to show that
$$p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r).$$

3. Use the double implication method to show that
$$\neg(q \vee r) \Longleftrightarrow (\neg p \wedge \neg q).$$

4. Use the double implication method to show that
$$\neg(q \wedge r) \Longleftrightarrow (\neg p \vee \neg q).$$

5. Show that $[(p \Rightarrow q) \Rightarrow r] \Longrightarrow [p \Rightarrow (q \Rightarrow r)]$.

6. Show that $[(p \Rightarrow q) \Rightarrow r] \Longrightarrow [p \Rightarrow (q \Rightarrow r)]$.

7. Show that $[(p \wedge \neg r) \vee (q \wedge \neg p) \vee (r \vee \neg q)] \Longleftrightarrow [(p \wedge \neg q) \vee (q \wedge \neg r) \vee (r \vee \neg p)]$.

8. Let p_n be the statement $2 + 4 + 6 + 8 + \cdots + (2n) = n(n+1)$. What is p_1? What is p_2. What is p_{10}? Which of them are true?

9. Let p_n be the statement $2 + 4 + 6 + 8 + \cdots (2n) = n(n+1)$. Show that p_n is true for all n by induction.

10. Let n be a natural number and let p_n be the statement "The number $n^3 - n$ is evenly divisible by 3".

    What is statement p_2?

    What is statement p_5?

    Are they both true?

11. Let n be a natural number and let p_n be the statement "The number $n^3 + 2n$" is evenly divisible by 3.

    Show that p_n is true for all n by induction.

12. Let p_n be the statement "In a party with n people, if every guest shakes hands once with every other guest, there must be $n(n-1)/2$ handshakes during the party."

    What is statement p_5? It is true?

    Show all statements p_n are true by induction for $n > 0$.

13. Prove by induction that $2n^3 + 3n^2 + n$ is always evenly divisible by 6.

14. Show by induction that $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$

15. Consider the statement p_n defined by

    $$1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} + \frac{n(n-1)(n-2)(n-3)}{24} = 2^n$$

    Check the truth of p_n for $n = 0$, $n = 1$, $n = 2$, $n = 3$, and $n = 4$.

    What can you conclude from this about the truth of p_n for all n?

16. Consider a collection of n straight lines drawn haphazardly in the plane, so that no two are parallel and no three or more intersect in a single point. How many points of intersection does this collection of n lines have? Note, the intersection point still exists even if it is off the paper.

    Consider small cases, and look for a formula. Prove your formula using induction.

17. Use induction on m to show that the number of functions from n-set to an m-set is m^n for $m \geq 1$.

18. Use induction on n to show that

    $$p \vee (q_1 \wedge q_2 \wedge \cdots \wedge q_n) = (p \vee q_1) \wedge (p \vee q_2) \wedge \cdots \wedge (p \vee q_n).$$

19. Use induction on n to show that

    $$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$$

20. Use induction on n to show that

    $$\neg(B_1 \cap B_2 \cap \cdots \cap B_n) = (\neg B_1) \cup (\neg B_2) \cup \cdots \cup (\neg B_n).$$