Exercises for Lectures 5 and 6

Cardinality of Finite Sets:

Lectures 3 and 4 we defined the cartesian product of two sets and discussed the cardinality of finite sets.

Among other things that

$$|\mathcal{P}(A)| = 2^{|A|} \qquad |A \times B| = |A| \cdot |B| \qquad |\mathcal{P}_k(A)| = \binom{|A|}{k} = \frac{|A|!}{k!(|A|-k)!}$$

We discussed encoding the subsets of A by bit vectors, and the connection to binary numbers.

We discussed the the inclusion/exclusion principle.

$$\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \\
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
|A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| \\
&\quad -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
&\quad +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap C| \\
&\quad -|A \cap B \cap C \cap D|
\end{aligned}$$

We showed the relation $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ and used it to construct Pascal's Triangle.

1. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Construct the sets $A \times B$, $\mathcal{P}(A) \times (B \times A)$, and $\mathcal{P}_2(A) \times \mathcal{P}_2(B)$.

2. Let $X = \{1, 2, \ldots 20\}$ and $B = \{1, 2, \ldots, 30\}$. Compute $|X \times Y|$, $|\mathcal{P}(X) \times (Y \times X)|$, and $|\mathcal{P}_2(X) \times \mathcal{P}_2(Y)|$.

3. Let X, Y, and Z be finite sets. Show that $|X \times (Y \times Z)| = |Z \times (Y \times X)|$.

4. Let A and B be finite sets, and suppose the $|A \cup B| = 100$ and $|A \cap B| = 50$. What is the largest value that $|A|$ can have? What can you say about B in this case?

   What is the smallest value that $|A|$ can have? What can you say about B in this case?

5. Suppose $|\mathcal{P}(A)| = |A \times A|$. What can you say about A.

6. Suppose $|\mathcal{P}(A)| = 4|A|$. What can you say about A.

7. Give an example of a set X with a subset $Y \subseteq X$ such that $|\mathcal{P}(Y) = X|$.

8. How many 10 digit numbers (5 = 0000000005 is also a 10 digit number) are either divisible by 1000 or divisible by 2?

9. How many 8 letter words either start with three vowels (a, e, i, o, u) or end with two vowels?

10. How many 9 letter words either start with 5 identical characters, like xxxxxfyry, end with 5 identical characters, like rorooooooo, or repeat in groups of three, like blablabla.

11. How many 7 letter words either start with x, end with x, or read the same forwards or backwards.

12. How may 6 letter words either start with two identical characters, end with two e's, read the same forwards or backwards, or repeat after 3 letters.