Ma2201/CS2022

Discrete Mathematics

D Term, 2013

Quiz 0111

Solutions

1. (**2 points**) Consider $f(n) = \dfrac{n^n}{n!}$. Circle the best answer.

a) $f(n) \approx 3^n$ b) $f(n) = e^n$.

c) $f(n)$ grows faster than 2^n d) none of the above

By Stirling's formula $f(n) \approx \frac{e^n}{2\pi n}$,

a) is not true. $f(n) \approx 3^n$ *means that* $\lim_{n\to\infty} \dfrac{f(n)}{3^n} = 1$, *but* $\lim_{n\to\infty} \dfrac{f(n)}{3^n} = \infty$.

b) is false, just check at $n = 1$.

c) is true since $e/2 > 1$ *and* $f(n)/2^n \approx \frac{(e/2)^n}{2\pi n}$.

d) false since c) is true.

2. (**2 points**) Which of the following numbers is closest in value to

$$\binom{100}{100} + \binom{101}{100} + \binom{102}{100} + \binom{103}{100} + \binom{104}{100} + \binom{105}{100} + \binom{106}{100} + \binom{107}{100}.$$

Circle the best answer.

a) 2^{105} b) $\dbinom{108}{100}$.

c) $\dbinom{108}{7}$ d) $\dbinom{108}{99}$

By the "hockey stick" formula, the eight terms sum to $\binom{108}{101}$ *which is equal to* $\binom{108}{108-101} = \binom{108}{7}$.

So c) is the best answer.

3. (**6 points**) Prove that for all n

$$\sum_{i=0}^{n} \binom{n}{i}\binom{2n}{n+i} = \binom{3n}{n}$$

The term on the right is the number of ways to choose n *things from* $3n$ *things. There terms on the right have two factors, the ways of choosing things from an* n*-set and a* $2n$*-set respectively. This suggests the following argument:*

Proof: Let A *be a set with* $|A| = n$ *and let* B *be a subset with* $|B| = 2n$, *and suppose* $A \cap B = \emptyset$, *so* $|A \cup B| = 3n$. *(A are the "golden elements")*

$\binom{3n}{n}$ *is the number of subsets of* $A \cup B$ *with cardinality* n.

For i, $0 \le i \le n$, $\binom{n}{i}$ *is the number of subsets of* A *with* i *elements and* $\binom{2n}{n-i}$ *is the number of subsets of* B *with* $n - i$ *elements so* $\binom{n}{i}\binom{2n}{n-i}$ *is the number of subsets of* $A \cup B$ *of cardinality* n *with* i *elements in* A *and the rest of the elements in* B. *Since the subsets of* $A \cup B$ *have from* 0 *to* n *elements in* A,

$$\sum_{i=0}^{n} \binom{n}{i}\binom{2n}{n-i} = \binom{3n}{n}.$$

Finally, by symmetry, $\binom{2n}{n-i} = \binom{2n}{2n-(n-i)} = \binom{2n}{n+i}$, *giving the result.*