Ma2201/CS2022 Discrete Mathematics D Term, 2013
Quiz 0110 Solutions

DOUBLE DOWN QUIZ:

◯ Normal ◯ Double Down ◯ TRIPLE DOWN

1. (**4 pts**) How many 144 letter strings of lower case letters $L = \{a, b, c, \ldots, z\}$ have either only vowels ($V = \{a, e, i, o, u\}$) in the the odd positions, start with 100 identical letters, or read the same forwards as backwards.

Let A be the strings with vowels in the odd positions, B the strings which start with 100 identical letters, and C be the strings which read the same forwards and backwards.

We want to compute $A \cup B \cup C$ and use inclusion exclusion.

$|A| = 5^{72} \cdot 26^{72}$*, since there are 72 positions with vowels and 72 having any letter.*

$|B| = 26^{45}$*, since the choice of the first letter forces the next 99, leaving 44 to be independently chosen.*

$|C| = 26^{72}$*, only the first 72 can be independently chosen.*

$|A \cap B| = 5^{23}26^{22}$*, since the first 100 letters must all be identical vowels, and the last 44 letters are half vowel, half normal.*

$|A \cap C| = 5^{72}$*, if it read the same forwards and backwards, there must be vowels in all positions, and the first 72 determine the rest.*

$|B \cap C| = 26$*, since all the letters must be the same, and*

$|A \cap B \cap C| = 5$*, since all the letters must be the same vowel.*

So

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 5^{72} \cdot 26^{72} + 26^{45} + 26^{72} - 5^{23} \cdot 26^{22} - 5^{72} - 26 + 5 \end{aligned}$$

2. (**4 points**) Let $X = \{0, 1, 2, \ldots, 100\}$, $Y = \{1, 2, 3, \ldots, 100\}$, and let $Z = \mathcal{P}(Y)$. Define a function $f : Z \longrightarrow X$ by $f(A) = |A|$.

Circle each of the following which must be true.

a) f is one-to-one b) f is onto

c) $|\{z \in Z \mid f(z) = 50\}| = 50^{100}$ d) $|\{z \in Z \mid f(z) = 50\}| = \binom{100}{50}$

a) is false, since $f(\{1, 2\}) = f(\{2, 3\}) = 2$.

b) is true, since a set with 100 elements has subsets of cardinality that can be anything from 0 to 100, for example the subsets: $|\emptyset| = 0$, $|\{1\}| = 1$, $|\{1, 2\}| = 2$, ldots, $|\{1, 2, 3, \ldots, 100\}| = 100$.

c) is false and d) is true: $f(z) = 50$ when the subset z has cardinality 50, of which there or 100 choose 50 possibilities.

3. (**2 points**) Let $A = \{a, b, c, d\}$ and $B = \{b, c\}$, and let C be the set of all functions with domain set B and target set A.

Compute the number of onto functions with domain set C and target set A.

(You can write your answer as an algebraic expression.)

$|C| = |A|^{|B|} = 4^2 = 16$.

The onto functions from a set with 16 elements to a set with 4 elements:
Use inclusion exclusion to compute the functions which are not *onto:*
Set F_x those functions for which x is not in the image, $x \in A$.
So we want

$$4^{16} - 4 \cdot 3^{16} + \binom{4}{2} \cdot 2^{16} + \binom{4}{3} \cdot 1^{16} - \binom{4}{4} \cdot 0^{16} = \sum_{i=0}^{4} (-1)^i \binom{4}{i} \cdot i^{16}$$