Ma2201/CS2022 Quiz 0110

Discrete Mathematics

D Term, 2012

PRINT NAME: ________________

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1. (4 pts) Let F_n denotes the Fibonacci sequence, so F_n satisfies $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.

Suppose $Q_0 = 2$, $Q_1 = 4$ and $Q_{n+2} = Q_{n+1} + Q_n$ for all $n \geq 0$.

Label the following True or False

a) ________ Q_n is uniquely defined for all $n \geq 0$.

b) ________ $Q_n = 2F_{n+2}$ for all $n \geq 0$.

c) ________ $Q_n = 2 + 2n$ for all $n \geq 0$.

d) ________ $Q_n = F_n + F_{n+3}$ for all $n \geq 0$.

Solution: a) is true. The first two terms are uniquely defined. Given that the first n items of the sequence are uniquely defined, the recurrence uniquely defines the next item in terms of the previous two items, extending the list of uniquely defined terms by one, and so by induction all the terms are uniquely defined.

b) is true. $2F_{n+2}$ satisfies the same recurrence:

$$2F_{n+2} + 2F_{(n+1)+2} = 2F_{n+2} + 2F_{n+3} = 2F_{n+4} = 2F_{(n+2)+2}$$

and satisfies the initial conditions: $2F_{0+2} = 2F_2 = 2 \cdot 1 = 2$, and $2F_{1+2} = 2F_3 = 2 \cdot 2 = 4$. So $Q_n = 2F_{n+2}$.

c) is false. $2 + 2n$ agrees for the first couple terms with Q_n, but does not satisfy the recurrence.

$[2 + 2n] + [2 + 2(n + 1)] = 6 + 4n = 2 + 2(n + 2) + 2n \neq 2 + 2(n + 2)$

d) is true. $F_n + F_{n+3}$ satisfies the same recurrence:

$$[F_n + F_{n+3}] + [F_{n+1} + F_{n+4}] = [F_n + F_{n+1}] + [F_{n+3} + F_{n+4}] = F_{n+2} + F_{n+5}$$

and satisfies the same initial conditions: $F_0 + F_3 = 0 + 2 = 2$ and $F_1 + F_4 = 1 + 3 = 4$.

2. (6 pts) Suppose the sequence $\{Z_n\}$ satisfies the recurrence

$$Z_{n+2} = 3Z_{n+1} - 2Z_n$$

and that $Z_0 = 5$ and $Z_1 = 6$.

Find Z_{2012}.

Solution: As we did for the Fibonacci sequence, looking for an exponential solution $Z_n = r^n$ yields

$$r^{n+2} = 3r^{n+1} - 2r^n$$

or $0 = r^n(r^2 - 3r + 2) = r^n(r - 1)(r - 2)$. Setting $r = 0$ just gives a sequence of 0's, incompatible with the initial conditions, so we have either $r = 1$ or $r = 2$. Neither of these gives $Z_0 = 5$ and $Z_1 = 6$, so again following our method for the Fibonacci sequence, we combine them

$$a \cdot 1^n + b \cdot 2^n = a + b \cdot 2^n$$

and solve for a and b.

We have $a + b \cdot 2^0 = a + b = 5$ and $a + b \cdot 2^1 = a + 2b = 6$.

Subtracting gives $b = 1$, and so $a = 4$, and so the sequence would yield $4 + 2^n$ satisfying the initial conditions, and just to be sure, checking the recursion,

$$3(4 + 2^{n+1}) - 2(4 + 2^n) = 12 + 3 \cdot 2^{n+1} - 8 - 2^{n+1} = 4 + 2 \cdot 2^{n+1} = 4 + 2^{n+2}$$

as required.

So $Z_{2012} = 4 + 2^{2012}$.