Ma2201/CS2022

Discrete Mathematics

D Term, 2011

Final Exam

Print Name: ____________________

Good Luck

Sign: ____________________

Choose any five of the following.

1. (**6 pts**) How many strings of six characters either start and end with the same letter, such as "ANDREA", or read the same forwards and backwards, such as "YTAATY", or contain only letters from the set $\{A, E, I, O, U\}$, such as "OOAIUO".

Solution: Let A be the subset of words with the same first and last letter, B the subset of words reading the same backwards and forwards, and C be the subset of words containing only vowels. We want to compute $A \cup B \cup C$. Note $B \subseteq A$.

$|A| = 26^5$, $|B| = 26^3$, $|C| = 5^6$, $|A \cap B| = |B| = 26^3$, $|A \cap C| = 5^5$, $|B \cap C| = 5^3$, and $|A \cap B \cap C| = |B \cap C| = 5^3$.

So by inclusion exclusion $|A \cup B \cup C| = 26^5 + 26^3 + 5^6 - 26^3 - 5^5 - 5^3 + 5^3 = 26^5 + 5^6 - 5^5$.

2. (**6 pts.**) Prove by induction that $n^3 \equiv n \bmod 3$.
(Be sure that your response is clear, neat and well written.)

Solution: Induction on n. For for the base case $n = 0$, we have $0^3 = 0 \equiv 0 \bmod 3$.

Induction step: Assume the statement is true for n, so we assume $n^3 \equiv n \bmod 3$ for some particular value of n. Now consider $(n+1)^3$.

$$
\begin{aligned}
(n+1)^3 &= n^3 + 3n^2 + 3n + 1 && \text{by the binomial theorem} \\
&\equiv n^3 + 1 \bmod 3 && \text{since } 3n^2 + 3n \text{ is divisible by } 3 \\
&\equiv n + 1 \bmod 3 && \text{since } n^3 \equiv n \bmod 3 \text{ by the induction hypothesis.}
\end{aligned}
$$

which is the statement for $n + 1$, as required.

3. (**6 pts.**) How many integers are there which are not divisible by any prime larger than 64 and not divisible by the cube of any prime.
Show all work.

Solution: How many primes are less than 64, to determine that we need only check the numbers less than 64 for divisibility by primes less than $\sqrt{64} = 8$, so we only have to check divisibility by 2, 3, 5, and 7.

This gives us a set $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61\}$ of 18 primes. The condition says that in the prime decomposition of the number only these primes occur and the exponent will be either 0, 1, or 2. By unique factorization the choices of exponent are independent, so there are $18^3 = 5,832$ such numbers.

4. (**6 pts.**) Compute the greatest common divisor, d, of 100 and 254 using the Euclidean algorithm. Then find integers n and m so that $d = n \cdot 100 + m \cdot 254$.
Show all steps.

$$\begin{aligned}
254 &= 2 \cdot 100 + 54 & 2 &= 13 \cdot (254 - 2 \cdot 100) - 7 \cdot 100 = 13 \cdot 254 - 33 \cdot 100 \\
100 &= 1 \cdot 54 + 46 & 2 &= 6 \cdot 54 - 7 \cdot (100 - 1 \cdot 54) = 13 \cdot 54 - 7 \cdot 100 \\
54 &= 1 \cdot 46 + 8 & 2 &= 6 \cdot (54 - 1 \cdot 46) - 1 \cdot 46 = 6 \cdot 54 - 7 \cdot 46 \\
46 &= 5 \cdot 8 + 6 & 2 &= 8 - 1 \cdot (46 - 5 \cdot 8) = 6 \cdot 8 - 1 \cdot 46 \\
8 &= 1 \cdot 6 + 2 & 2 &= 8 - 1 \cdot 6 \\
6 &= 3 \cdot 2 + 0 & & \text{so 2 is the greatest common divisor. Now work back up}
\end{aligned}$$

We get $= 13 \cdot 254 - 33 \cdot 100$.

5. (**6 pts.**) Assume that the sequence $\{a_0, a_1, a_2, \ldots\}$ satisfies the recursion

$$a_{n+1} = a_n + 2a_{n-1}.$$

We know that $a_0 = 4$ and $a_2 = 13$.

What is a_5?

Show all work.

The recursion gives the quadratic $r^2 - r - 2 = (r-2)(r+1)$, so we look for a solution of the form $a2^n + b(-1)^n$.

Since $a_0 = 4$ we have $4 = a + b$.

Since $a_2 = 13$ we have $13 = 4a + b$.

Subtracting them give $9 = 3a$, or $a = 3$, so $b = 1$.

So $a_n = 3 \cdot 2^n + (-1)^n$ and $a_5 = 3 \cdot 32 - 1 = 95$.

6. (**6 pts.**) In how many ways can you read off the word MATHEMATICS from the following table by starting at the top right corner and moving successively one step right or one step down.

M	A	T	H	E	M
A	T	H	E	M	A
T	H	E	M	A	T
H	E	M	A	T	I
E	M	A	T	I	C
M	A	T	I	C	S

There are 10 letters in mathematics, and each move is either right or down. There must be 5 rights and 5 downs, so there are $\binom{10}{5}$ ways to choose the 5 rights among the 10 moves.

Or we can move from the top left and fill in the grid with the number of ways to start the word mathematics. and at the given square. It will look just like what we did for Pascal's triangle.

1	1	1	1	1	1
1	2	3	4	5	6
1	3	6	10	15	21
1	4	10	20	35	56
1	5	15	35	70	126
1	6	21	56	126	252

in other words $\binom{10}{5}$ ways.