Homework 3

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Section 10.4

Question 13

Use the limit comparison test to determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$$

We can use the limit comparison test to compare $a_n = \frac{5^n}{\sqrt{n}\,4^n}$ to a known divergent series $b_n = \frac{1}{\sqrt{n}}$.

$$\begin{aligned} c &= \lim_{n\to\infty} \frac{a_n}{b_n} \\ &= \lim_{n\to\infty} \frac{5^n}{4^n} \\ &= \lim_{n\to\infty} \left(\frac{5}{4}\right)^n \\ &= \boxed{\infty} \end{aligned}$$

Since $c = \infty$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$ also diverges.

Question 18

Find if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$$

We can use the limit comparison test by comparing $a_n = \frac{3}{n+\sqrt{n}}$ to a known divergent series $b_n = \frac{3}{n}$.

$$\begin{aligned}
c &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{3}{n+\sqrt{n}} \cdot \frac{n}{3} \\
&= \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} \\
c &= 1 > 0
\end{aligned}$$

Since $c > 0$, and $\sum_{n=1}^{\infty} \frac{3}{n}$ is a known divergent series (3 times the harmonic series), $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ also diverges.

Question 42

Determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$$

We can determine if this series converges by using the limit comparison test with $a_n = \frac{\ln n}{\sqrt{n}\, e^n}$ and comparing to $b_n = \frac{1}{e^n}$. b_n is a geometric series with $r = \frac{1}{e} < 1$, so we know that it converges. Therefore, if $c = \frac{a_n}{b_n} \geq 0$, we will know that $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$ converges.

$$\begin{aligned}
c &= \lim_{n\to\infty} \frac{a_n}{b_n} \\
&= \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}\, e^n} \cdot \frac{e^n}{1} \\
&= \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}
\end{aligned}$$

We can evaluate this limit using L'Hopital's rule.

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}} &= \lim_{n\to\infty}\frac{\frac{1}{n}}{-\frac{1}{2}n^{-\frac{1}{2}}} \\
&= \lim_{n\to\infty}\frac{1}{n}\cdot-\frac{n^{\frac{1}{2}}}{1} \\
&= \lim_{n\to\infty}-\frac{1}{n^{\frac{1}{2}}} \\
&= \boxed{0}
\end{aligned}$$

Since $c = 0$ and $\frac{1}{e^n}$ is a known convergent series, $\underline{\sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{n}\,e^n}\text{ converges}}$.

Section 10.5

Question 18

Determine if the following series converges or diverges

$$\sum_{n=1}^{\infty}(-1)^n\,n^2e^{-n}$$

We will begin by seeing if $\sum_{n=1}^{\infty}|a_n|$ is convergent where $a_n = (-1)^n\,n^2e^{-n}$ by using the ratio test. If $\sum_{n=1}^{\infty}|a_n|$ is convergent, then the original series would be absolutely convergent.

$$\begin{aligned}
\sum_{n=1}^{\infty}a_n &= \sum_{n=1}^{\infty}(-1)^n\,n^2e^{-n} \\
\sum_{n=1}^{\infty}|a_n| &= \sum_{n=1}^{\infty}n^2e^{-n} \\
p &= \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{e^{n+1}}\cdot\frac{e^n}{n^2} \\
&= \lim_{n\to\infty}\frac{(n+1)^2}{en^2} \\
&= \lim_{n\to\infty}\frac{n^2+2n+1}{en^2} \\
p &= \frac{1}{e} < 1
\end{aligned}$$

Since $p < 1$ for $\sum_{n=1}^{\infty}|a_n|$, $\sum_{n=1}^{\infty}(-1)^n\,n^2e^{-n}$ is absolutely convergent
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Question 19

Determine if the following series converges or diverges

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{n!}{e^n}\right)$$

Since the series is an alternating series, we can check if the series passes the alternating series test with $u_n = \frac{n!}{e^n}$. We will begin by testing if u_n is decreasing for all integers n by solving for $\frac{u_{n+1}}{u_n}$, where if $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} > 1$ for any integer n, $\lim_{n\to\infty} n > n+1$ and the series would not be eventually decreasing and therefore would not pass the alternating series test.

$$\begin{aligned}
\frac{u_{n+1}}{u_n} &= \frac{(n+1)!}{e^{n+1}}\cdot\frac{e^n}{n!} \\
&= \frac{n+1}{e} \\
\lim_{n\to\infty}\frac{u_{n+1}}{u_n} &= \lim_{n\to\infty}\frac{n+1}{e} \\
&= \boxed{\infty}
\end{aligned}$$

Since $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} > 1$, u_n is not eventually decreasing, meaning the series does not pass the alternating series test. Therefore, $\underline{\sum_{n=1}^{\infty}(-1)^n\left(\frac{n!}{e^n}\right)\text{ diverges.}}$

Question 26

Determine if the following series converges or diverges

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{3n}\right)^{n}$$

We can use the n^{th} test for divergence to determine if the series diverges by finding $\lim_{n\to\infty} a_n$ with $a_n = \left(1-\frac{1}{3n}\right)^n$. If this limit does not approach 0, the series diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^{n}$$

This limit yields the indeterminate form 1^{∞}. We can evaluate this limit by taking the logarithm of the limit and exponentiating the result. We can then use the power rule of logarithms to evaluate the limit.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^{n}\\ &= e^{\lim\limits_{n\to\infty} \ln\left(1-\frac{1}{3n}\right)^{n}}\\ &= e^{\lim\limits_{n\to\infty} n\ln\left(1-\frac{1}{3n}\right)}\end{aligned}$$

Evaluating this limit yields the indeterminate form $0\cdot\infty$. So, we will manipulate this limit to a form where L'Hopital's rule can be applied, and then evaluate the limit using L'Hopital's rule.

$$\begin{aligned}\lim_{n\to\infty} a_n &= e^{\lim\limits_{n\to\infty} n\ln\left(1-\frac{1}{3n}\right)}\\ &= e^{\lim\limits_{n\to\infty} \frac{\ln\left(1-\frac{1}{3n}\right)}{\frac{1}{n}}}\\ &= e^{\lim\limits_{n\to\infty} \frac{\frac{1}{1-\frac{1}{3n}}\cdot\frac{1}{3}n^{-2}}{-n^{-2}}}\\ &= e^{\lim\limits_{n\to\infty} -\frac{1}{3}\left(\frac{1}{1-\frac{1}{3n}}\right)}\\ \lim_{n\to\infty} a_n &= \boxed{e^{-\frac{1}{3}} > 0}\end{aligned}$$

Since $\lim_{n\to\infty} a_n > 0$, the series fails the n^{th} term test. Therefore, $\sum_{n=1}^{\infty}\left(1-\frac{1}{3n}\right)^n$ diverges.

Question 36

Determine if the following series converges

$$\sum_{n=1}^{\infty} \frac{n\,2^n(n+1)!}{3^n\,n!}$$

We can determine if this sum converges using the ratio test and solving for $p = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ with $a_n = \frac{n\,2^n(n+1)!}{3^n\,n!}$. If $p < 1$, the ratio test will show that the series converges.

$$\begin{aligned}
a_n &= \frac{n\,2^n(n+1)!}{3^n\,n!} \\
a_{n+1} &= \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}\,(n+1)!} \\
p &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}\,(n+1)!} \cdot \frac{3^n\,n!}{n\,2^n(n+1)!} \\
&= \lim_{n\to\infty} \frac{n+2}{3n} \\
p &= \boxed{\frac{1}{3} < 1}
\end{aligned}$$

Since $p = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, the series $\sum_{n=1}^{\infty} \frac{n\,2^n(n+1)!}{3^n\,n!}$ converges by the ratio test.