WORCESTER POLYTECHNIC INSTITUTE

Tenth ANNUAL INVITATIONAL MATH MEET

OCTOBER 23, 1997

TEAM EXAM QUESTION SHEET

DIRECTIONS : Please write your answers on the Team Answer Sheet provided. This part of the contest is 30 minutes. Each correct answer to questions 1-14 is worth 3 points. Calculators MAY NOT be used.

1 Find real $x > 0$ such that $(4x)^{\log_6 4} = (5x)^{\log_6 5}$.

2 Determine the coordinates of the point of tangency of the following two spheres:

$$x^2 + y^2 + z^2 + 2x - 10y - 6z + 34 = 0, \quad x^2 + y^2 + z^2 + 6x - 6y - 8z + 30 = 0.$$

3 The following figures show three views of a *nonstandard* die. How many dots are on the bottom face in Figure 1?

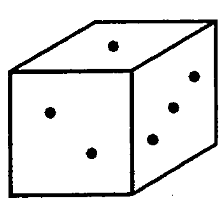


Figure 1

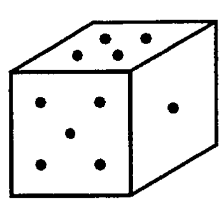


Figure 2

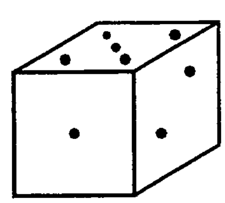


Figure 3

4 How many roots does the following equation

$$4\sin x - 4\cos^2 x = x^2/16 - 5$$

have?

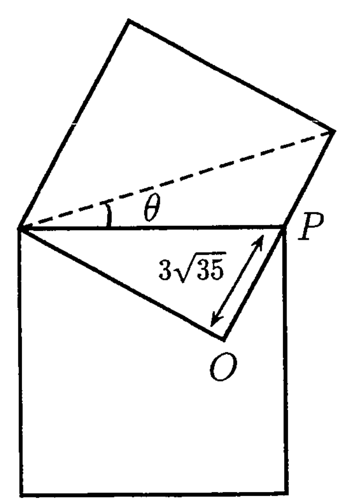
5 If the vertical line $x = m$ divides the triangle with vertices (0,0), (2,3) and (14,3) in the XY-plane into two regions of equal area, determine m.

6 A codeword is formed using 3 of the 26 letters in the English alphabet. The codeword must begin with the letter **A** and end with one of the 13 letters **N** to **Z**. Assuming no letter can be used more than once, how many such codewords are there?

7 How many zeroes are at the end of the expansion of 62! ?

8 A cylindrical pipe lies horizontally along the ground. Suppose the pipe is 50 ft long and has an interior diameter of 12 ft. If the water in the pipe is 9 ft deep at the center, how much water is in the pipe?

9 Two squares are placed (see the following figure) such that $OP = 3\sqrt{35}$. Let θ be the angle between the diagonal of the smaller square and an edge of the larger square. What is the sum of the areas of the two squares, if $\tan\theta = \dfrac{29 - 3\sqrt{35}}{29 + 3\sqrt{35}}$?



10 Let $[x]$ denote the greatest integer *not* exceeding the real numbers x. Find the set of all real numbers x which satisfy the simultaneous equations:

$$y = 2[x] + 3, \qquad y = 3[x-1] + 2.$$

11 A certain five-digit number has the property that with a 1 after it, it is three times as large as with a 1 before it. Find the number.

12 A contest is conducted as follows:

- You are to choose an initial distribution of 50 white and 50 black pearls into two identical boxes. You may choose any distribution at all. For example, you may put all 100 in one box and none in the other; or you may put 17 white and 12 black pearls in one box and the rest in the other.
- You will be blindfolded, the positions of the two boxes will be shuffled, and you will be asked to choose one of the boxes.
- Finally, you will select one pearl from the chosen box.

If you select a black pearl, you win \$1,000,000. If not, you win nothing. Suppose you should put w white pearls and b black pearls in one box (and the rest in the other box). Which of the following choice(s) for the pair (w, b) would correspond to your greatest chance of winning the money?

(a) (50,0) (b) (25,25) (c) (0,1) (d) None of (a), (b), (c).

13 The circumference of an inscribed n-gon of a given circle measures a inches and the circumference of a circumscribed $2n$-gon around the same circle measures b inches. Calculate the circumference of a $2n$-gon inscribed in this circle in terms of a and b. (Assume all polygons are regular.)

14 Suppose 27 unit cubes (each with side length 1 foot) are stacked together to form a 3 ft $\times$ 3ft $\times$ 3ft cube C. If you are free to choose and remove any two of the 27 cubes, what would be the MAXIMUM increase in the *exposed surface area* of the remaining solid?