exercise 1:
Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$, if $x \leq 0$, $f(x) = \exp(-\frac{1}{x})$, if $x > 0$. Show that f is in $C^\infty(\mathbb{R})$.

Hint: Show by induction on k that f is in $C^k(\mathbb{R})$ and that for $f^{(k)}(x) = \dfrac{P_k(x)}{Q_k(x)} \exp(-\dfrac{1}{x})$ if $x > 0$, where P_k and Q_k are polynomials.

exercise 2:
Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two intervals in $\mathbb{R}$ such that $a_1 < a_2$ and $b_1 < b_2$. (i). Explain why if we want compute $1_A * 1_B$, without loss of generality we may assume that $a_2 - a_1 \leq b_2 - b_1$.
(ii). Compute and graph $1_A * 1_B$.

exercise 3:
Define

$$g(x) = e^{-\frac{1}{x+1} - \frac{1}{1-x}} \text{ if } -1 < x < 1, \quad 0 \text{ otherwise },$$

$I = \int_{-1}^{1} g$, and $g_n(x) = g(nx)nI^{-1}$. Note that g is in $C^\infty(\mathbb{R})$. Then fix $\alpha > 0$ and define f to be continuous from $\mathbb{R}$ to $\mathbb{R}$ and such that $f(x) = 0$ if $x \leq -1 - \alpha$ or $x \geq 1 + \alpha$, $f(x) = 1$ if $-1 \leq x \leq 1$, and f is linear plus a constant on $[-1 - \alpha, -1]$ and on $[1, 1 + \alpha]$.
Set $f_n(x) = \int_{-\infty}^{\infty} g_n(x - t) f(t) dt$.

(i). Show that $f_n(x) = \int_{x - \frac{1}{n}}^{x + \frac{1}{n}} g_n(x - t) f(t) dt$ and that f_n is even.
(ii). Show that $f_n(x) = 0$ if $x \leq -1 - \alpha - \frac{1}{n}$ or $x \geq 1 + \alpha + \frac{1}{n}$.
(iii). If $\frac{1}{n} < 1$ show that $f_n(x) = 1$ if $-1 + \frac{1}{n} \leq x \leq 1 - \frac{1}{n}$.
(iv). Show that f_n is increasing in $[-1 - \alpha - \frac{1}{n}, -1 + \frac{1}{n}]$.