exercise 1:
Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) < +\infty$.
(i). Let f be in $L^\infty(X)$. Show that for all $p \geq 1$, $|f|^p$ is in $L^1(X)$, and $\limsup_{p\to\infty}(\int |f|^p)^{1/p} \leq \|f\|_\infty$.
(ii). Let ϵ be positive. What can be said about the set $A = \{x \in X : |f(x)| \geq \|f\|_\infty - \epsilon\}$? Infer that $\lim_{p\to\infty}(\int |f|^p)^{1/p} = \|f\|_\infty$.

exercise 2:
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f_n be a sequence of measurable functions on X and f a measurable function on X. We say that f_n converges to f in measure if for every positive ϵ,

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\})$$

converges to zero.
Show that if f_n converges to f in L^1 norm, then f_n converges to f in measure. Is the converse true?

exercise 3:
From Royden's textbook, 4.4.29.

exercise 4:
From Browder's textbook, 10.9.7.

Exercise 5:
Let $f \in L^1([0,1])$, with $f \geq 0$. Show that for any positive integer $f^{\frac{1}{n}}$ is in $L^1([0,1])$ and

$$\lim_{n\to\infty} \int_0^1 f^{\frac{1}{n}}(x)dx = m(\{x \in [0,1] \mid f(x) > 0\}),$$

where m is the Lebesgue measure.

Exercise 6:
Let (a, b) be an interval in $\mathbb{R}$ and f in $L^2((a, b))$. Assume that for every polynomial function P, $\int_a^b fP = 0$. Show that f is zero. **Hint**: Use the Weierstrass approximation theorem.

Exercise 7:
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f be in $L^1(X)$ and A_n a sequence in $\mathcal{A}$ such that $\mu(A_n)$ converges to zero. Show that $\int_{A_n} f$ converges to zero. **Hint:** In a first step, assume that f is a simple function.

exercise 8:

Let S be the square $[0,1] \times [0,1]$ and f be the function defined almost everywhere on S by the formula $f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$. Show by a direct calculation that $\int_S |f| = +\infty$.