Assignment 3

exercise 1:
Let X be a set and $\mathcal{A}$ be a σ- algebra of subsets of X. Let Y be a subset of X. Define a subset $\mathcal{B}$ of $\mathcal{P}(Y)$ by setting $\mathcal{B} := \{A \cap Y : A \in \mathcal{A}\}$. Show that $\mathcal{B}$ is a σ- algebra of subsets of Y.

exercise 2:
Let X be a set and $\mathcal{A}$ be a subset of $\mathcal{P}(X)$.
(i). If $\mathcal{A}$ is closed under complementation and set difference, show that $\mathcal{A}$ is closed under finite union and finite intersection.
(ii). If $\mathcal{A}$ contains X and is closed under set difference and finite intersection, show that $\mathcal{A}$ is closed under finite union and set complementation.

exercise 3:
Let X be a set and $\mathcal{A}_i$ be a σ- algebra of subsets of X for each i in I. Show that $\bigcap\limits_{i \in I} \mathcal{A}_i$ is σ- algebra of subsets of X.

exercise 4:
(i). Let V be an open subset of $\mathbb{R}$ which is neither empty nor equal to $\mathbb{R}$. Let V^c be the complement of V. Show that

$$V = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : |x| \leq n \text{ and } d(x, V^c) \geq \frac{1}{n}\}$$

(ii). Infer that every non empty open subset of $\mathbb{R}$ is a countable union of compact subsets.
(iii). Infer that every open subset of $\mathbb{R}$ is a countable union of open intervals.

exercise 5:
Let E be a subset of $\mathbb{R}$. Show using, the outer measure m^* on $\mathbb{R}$ and the fact that it is sub-additive:
(i). If E is Lebesgue measurable, then E^c is Lebesgue measurable.
(ii). If $m^*(E) = 0$, then E is Lebesgue measurable.

exercise 6:
(i). Let $(X, \mathcal{A}, \mu)$ be a measure space, B_n a decreasing sequence of $\mathcal{A}$ such that $\mu(B_1) < \infty$. Show that $\lim\limits_{n\to\infty} \mu(B_n) = \mu(\bigcap\limits_{n=1}^{\infty} B_n)$.

(ii). Find a measure space $(X, \mathcal{A}, \mu)$ and a decreasing sequence B_n of $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(B_n) \neq \mu(\bigcap_{n=1}^{\infty} B_n)$.

exercise 7:

Let S be the subset of $[0,1]$ defined by all the sums of the series $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ where a_n is in the set $\{0,2,4,6,8\}$.

(i). Assume that $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \sum_{n=1}^{\infty} \frac{b_n}{10^n}$ where a_n and b_n are in $\{0,2,4,6,8\}$. Show that $a_n = b_n$ for all interger $n \geq 1$.

Hint: Show that

$$|\sum_{n=1}^{p} \frac{a_n}{10^n} - \sum_{n=1}^{p} \frac{b_n}{10^n}| \leq \frac{8}{9} \frac{1}{10^p}$$

(ii). Show that S is uncountable.

Hint: Define $f : S \to \mathcal{P}(\mathbb{N}^*)$: let $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$, $a_n \in \{0,2,4,6,8\}$. $p \notin f(x)$ if $a_p = 0$, $p \in f(x)$ otherwise. It suffices to show that f is surjective.

(iii). Show that S is Lebesgue measurable has Lebesgue measure zero.

Hint: It suffices to show that the outer measure of S is zero. Use the set S_p defined by all the sums of the series $\sum_{n=1}^{p} \frac{a_n}{10^n}$ where a_n is in the set $\{0,2,4,6,8\}$, and

$$T_p = \bigcup_{y \in S_p} [y - \frac{8}{9} 10^{-p}, y + \frac{8}{9} 10^{-p}].$$

exercise 8

Let X be a set and $\mathcal{A}$ a σ algebra of subsets of X. Let A be in $\mathcal{A}$ and f a function from X to $\overline{\mathbb{R}}$ such that its restrictions to A and to A^c are measurable. Prove that f is measurable.

exercise 9:

Let X be a set and f and g two functions from X to $\overline{\mathbb{R}}$ such that $f(x) + g(x)$ is well defined for all x in X. Show that for any t in $\mathbb{R}$

$$\{x \in X : f(x) + g(x) < t\} = \bigcup_{q \in \mathbb{Q}} \{x \in X : f(x) < t - q\} \cap \{x \in X : g(x) < q\}$$

Infer that if $(X, \mathcal{A}, \mu)$ is a measure space and f and g are measurable functions from X to $\overline{\mathbb{R}}$ such that $f(x) + g(x)$ is well defined for all x in X, then $f + g$ is measurable.

exercise 10:

Let u_n be a sequence valued in $\overline{\mathbb{R}}$. Using the definition of $\liminf$ and of $\limsup$ given in class, show that $\liminf u_n \leq \limsup u_n$. Show that u_n converges in $\overline{\mathbb{R}}$ if and only if $\liminf u_n \geq \limsup u_n$ (for brevity, only cover the case where $\liminf u_n$ and $\limsup u_n$ are both finite).