Assignment 2

exercise 1:

Let X, Y be two sets, A, A_2 two subsets of X, B, B_2 two subsets of Y, and $f : X \to Y$ a function. Prove or disprove:
(i). $f(f^{-1}(B)) \subset B$
(ii). $f(f^{-1}(B)) = B$
(iii). if $f(A) \subset B$, then $A \subset f^{-1}(B)$
(iv). $f^{-1}(B \cup B_2) = f^{-1}(B) \cup f^{-1}(B_2)$
(v). $f(A \cup A_2) = f(A) \cup f(A_2)$
(vi). $f(A \cap A_2) = f(A) \cap f(A_2)$

exercise 2:

(i). Let $f : [0, 1] \to \mathbb{R}$, $f(x) = \sqrt{x}$. Show that f is uniformly continuous but not Lipschitz continuous.
(ii). Let $g : (0, 1) \to \mathbb{R}$, $g(x) = \frac{1}{x}$. Show that g is not uniformly continuous.

exercise 3:

Let (X, ρ) be a metric space and x_n be a sequence in X. We proved in class that if $\sum_{n=1}^{\infty} \rho(x_{n+1}, x_n)$ converges, then x_n is a Cauchy sequence. Conversely, if x_n is a Cauchy sequence, does $\sum_{n=1}^{\infty} \rho(x_{n+1}, x_n)$ converge?

exercise 4:

Let L be a linear function between the two normed spaces (V_1, N_1) and (V_2, N_2). Show that the following conditions are equivalent:
(i). L is continuous at 0.
(ii). L is Lipschitz continuous.
(iii). $\exists C > 0$, $\forall x \in V_1$, $N_2(Lx) \leq C N_1(x)$.

exercise 5:

Let A and B be two subsets of $\mathbb{R}^d$.
(i). If A is closed and B is compact, show that $A + B$ is closed.
(ii). If A is closed and B is closed, is $A + B$ is closed?