Assignment 1

exercise 1:
Set for $x = (x_1, ..., x_d)$ in $\mathbb{R}^d$, $\|x\|_\infty = \max\{|x_i| : i = 1, .., d\}$. Show that $\| \|_\infty$ defines a norm on $\mathbb{R}^d$.

exercise 2:
The sets of sequences l^2 and l^∞ were introduced in class together with $\|u\|_2$ and $\|v\|_\infty$ for u in l^2 and v in l^∞. Show that $(l^2, \| \|_2)$ and $(l^\infty, \| \|_\infty)$ are normed spaces. You may use in your solution that $(\mathbb{R}^d, \| \|_2)$ and $(\mathbb{R}^d, \| \|_\infty)$ are normed spaces.

exercise 3:
Let x be in $\mathbb{R}^d$ and $p \geq 1$ a real number.
(i). Show that $|x_1|^p + ... + |x_d|^p \leq d\|x\|_\infty^p$.
(ii). Show that $\lim\limits_{p\to\infty} \|x\|_p = \|x\|_\infty$.

exercise 4:
Let x and y be in $\mathbb{R}^d$ with $y \neq 0$. Optimize the function $f(t) =< x + ty, x + ty >$, $t \in \mathbb{R}$, in order to prove the Cauchy Schwartz inequality.

exercise 5:
Let (X, ρ) be a metric space. Suppose that X is a finite set. Show that any subset of X is both open and closed.

exercise 6:
Let (X, ρ) be a metric space. Let A be a subset of X: (A, ρ) is also a metric space. Show that V is an open subset of A if and only if there is an open subset W of X such that $V = W \cap A$.
Suggestion: make sure to use distinct notations for open balls in X, $B_X(x, \alpha)$, and open balls in A, $B_A(a, \alpha)$.

exercise 7:
Let x_n be a convergent sequence in the metric space (X, ρ). Show that the limit of x_n is unique.

exercise 8:
Let (X, ρ) be a metric space.

(i). In class, we said that a subset A of X is bounded if there exists $M > 0$ such that for all x and y in A, $\rho(x,y) \leq M$. Show that A is bounded if and only if, $\forall z \in X$, $\exists M_2 > 0$, $\forall x \in A$, $\rho(x,z) \leq M_2$.
(ii). Let V_1 and V_2 be two bounded subsets of (X,ρ). Show that $V_1 \cup V_2$ is bounded.

<u>exercise 9</u>:
Prove that l^2 is complete.

<u>exercise 10</u>:
Let X be a metric space and $\mathrm{int}(S)$ the interior of any subset S. Prove or disprove.
(i). $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.
(ii). $\mathrm{int}(A \cup B) = \mathrm{int}(A) \cup \mathrm{int}(B)$.

<u>exercise 11</u>:
Let X be a metric space and A a subset of X. Show that A is closed if and only if

$$A = \{x \in X : \mathrm{dist}(x,A) = 0\}.$$