<u>exercise 1</u>:
Let V be a finite-dimensional vector space over $\mathbb{R}$ and A a symmetric operator in $\mathcal{L}(V)$. Let $n = \dim V$ and $\lambda_1 < ... < \lambda_p$ be the ordered eigenvalues of A.
(i). Show that $\lambda_p = \max\limits_{x\in V, \|x\|=1} < Ax, x >$.
(ii). Show that $\lambda_1 = \min\limits_{x\in V, \|x\|=1} < Ax, x >$.
(iii). Let $E_p = \text{Ker } (A - \lambda_p I)$. Show that $\lambda_{p-1} = \max\limits_{x\in E_p^{\perp}, \|x\|=1} < Ax, x >$.
(iv). If $\lambda_1 \geq 0$ find $\max\limits_{x\in V, \|x\|=1} \|Ax\|$.

<u>exercise 2</u>:
Find a complex square matrix M such that $M^T = M$ and M is not diagonalizable.

<u>exercise 3</u>:
Let V be a finite-dimensional vector space over $\mathbb{C}$ and A a Hermitian operator in $\mathcal{L}(V)$ such that its eigenvalues are in $[0, \infty)$. Show that the eigenvalues of A are equal to the square roots of the eigenvalues of A^*A.

<u>exercise 4</u>:
Let V be a finite-dimensional inner product space over $\mathbb{R}$ and U a unitary operator in $\mathcal{L}(V)$. Let M be the matrix of U in an orthonormal basis of V.
(i). Show that M is normal and that any complex eigenvalue λ of M satisfies $|\lambda| = 1$.
(ii). Show that if λ is an eigenvalue of M, then $\overline{\lambda}$ is an eigenvalue of M.
(iii). Prove that we can find an orthonormal basis of V such that the matrix of U in that basis is block diagonal, with blocks of size 1 containing -1, or 1, and blocks of size 2 in the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, for some θ in $\mathbb{R}$.

Hint: let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be an eigenvalue of M and v an eigenvector. Use $\frac{1}{\sqrt{2}}(v + \overline{v})$ and $\frac{1}{\sqrt{2}i}(v - \overline{v})$,

<u>exercise 5</u>:
Let $e_1, ..., e_n$ be the natural basis of K^n. Let $a_0, ..., a_{n-1}$ be n scalars in K and M in $K^{n\times n}$ such that $Me_j = e_{j+1}$, if $j = 1, ..., n-1$ and

$$Me_n = -a_0 e_1 - a_1 e_2 ... - a_{n-1} e_n.$$

(i). Write down the matrix M.
(ii). Find $M^j e_1$ for $j = 0, ..., n$.
(iii). Prove that the characteristic polynomial of M is $X^n + a_{n-1}X^{n-1} + ... + a_0$. **Hint :** it is insufficient to show that P(M) =0, since for $Q = (X-2)(X-1)^{n-1}$, $Q(I_n) = 0$.

exercise 6:
Exercise 9.2.2.

exercise 7:
Exercise 9.2.5.