exercise 1:
(i). Find a finite dimensional space V and A in $\mathcal{L}(V)$ such that V is not the sum of Ker A and Im A.
(ii). Is it possible to find such an example with the additional requirement that
Ker $A \cap$ Im A $= \{0\}$?

exercise 2:
Let R be a unitary matrix in $\mathbb{R}^{2\times 2}$ such that $\det R = 1$. Show that there is a θ in $[0, 2\pi]$ such that

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

exercise 3:
Let $V = \mathbb{R}^n$, $n \geq 3$, with its usual natural basis $e_1, ..., e_n$ and inner product. Define the cross product,

$$C : V^{n-1} \to V$$

$$C(v_1, ..., v_{n-1}) = \sum_{j=1}^{n} e_j \det(e_j, v_1, ..., v_{n-1}).$$

(i). Show that C is linear in each of its argument.
(ii). Show that the vectors $v_1, ..., v_{n-1}$ are dependent if and only if $C(v_1, ..., v_{n-1}) = 0$.
(iii). Let R be a rotation, that is, $R \in \mathbb{R}^{n\times n}$, $R^T R = I_n$ and $\det R = 1$. Show that $RC(v_1, ..., v_{n-1}) = C(Rv_1, ..., Rv_{n-1})$.

exercise 4:
Let V be a finite dimensional Hermitian space and A in $\mathcal{L}(V)$.
(i). Show that A and A^* have the same rank.
(ii). Do A and A^* have the same image?

exercise 5:
Let $V = C^\infty(0, 1)$, and T in $\mathcal{L}(V)$ defined by $Tf = f'$. Find all eigenvalues of T and corresponding eigenvectors.

exercise 6:
Let V be a vector space over $\mathbb{R}$ such that $\dim V = 2p$ where p is a positive integer. Show that there is a T in $\mathcal{L}(V)$ such that T has no eigenvalue.

<u>exercise 7</u>:
Let V be an n-dimensional vector space over K. Find a T in $\mathcal{L}(V)$ that is not diagonalizable.

<u>exercise 8</u>:
Let V be a vector space and $W, W_1, W_2, ..., W_p$ be subspaces. Assume that $V = W_1 \oplus W$ and $W = W_2 \oplus ... \oplus W_p$.
Show that $V = W_1 \oplus W_2 \oplus ... \oplus W_p$.

<u>exercise 9</u>:
Let n be in $\mathbb{N}$ and p, q in $\mathbb{N}$ greater or equal than 1 such that $p + q = n$.
(i). Define the matrix N in $K^{n\times n}$ by blocks

$$N = \begin{pmatrix} I_p & D \\ 0 & I_q \end{pmatrix}.$$

Find $\det N$.
Now let M be in $K^{n\times n}$

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $A \in K^{p\times p}$, $B \in K^{q\times q}$.
(ii). If A or B is singular, show that $\det M = 0$.
(iii). Show that $\det M = \det A \det B$. **Hint:** Use matrix block multiplication.