<u>exercise 1</u>:
Let $[a,b]$ be an interval in $\mathbb{R}$ and $x_0 < x_1 < ... < x_n$ be in $[a,b]$. Let $y_0, ..., y_n$ be in $\mathbb{R}$. Using Vandermonde determinants show that there is a unique polynomial P in $\mathbb{R}[X]$ with $\deg P \leq n$ such that $P(x_j) = y_j$, for $j = 0, ..., n$.

<u>exercise 2</u>:
Let A be in $K^{n\times n}$. Assume that A^T is singular. The goal is to prove that A is also singular.
(i). Let $R_1, ..., R_n$ be the rows of A. Show that there are $a_1, ..., a_n$ that are not all zero and such that $a_1 R_1 + ... + a_n R_n = 0$.
(ii). Infer that the image space of A is included in a hyperplane.
(iii). Infer that A is singular.

<u>exercise 3</u>:
Let V be a vector space and V_1, V_2 two subspaces such that $V = V_1 \bigoplus V_2$. Assume that $\{e_1, ..., e_m\}$ is a basis of V_1 and that $\{f_1, ..., f_n\}$ is a basis of V_2. Show that $\{e_1, ..., e_m, f_1, ..., f_n\}$ is a basis of V.

<u>exercise 4</u>:
Let V be a finite dimensional space and $\mathcal{B}$ a basis of V. Let $B : V \times V \to K$ be a bilinear form and M its matrix in $\mathcal{B}$. Show that B is symmetric if and only if M is symmetric.

<u>exercise 5</u>:
Let V be a finite dimensional space over $\mathbb{R}$ and $\mathcal{B}$ a basis of V. Let $B : V \times V \to \mathbb{R}$ be a bilinear form and M its matrix in $\mathcal{B}$. Assume that B is symmetric and non-negative, that is, $B(x,x) \geq 0$ for all x in V. Show that B is positive definite if and only if M is invertible.
Remark: positive definite means that $B(x,x) > 0$ for all non-null vectors x in V.

<u>exercise 6</u>:
Textbook exercise 5.3.2.

<u>exercise 7</u>:
Textbook exercise 5.5.2.

<u>exercise 8</u>:
Let $(V, <,>)$ be a Euclidean vector space over $\mathbb{R}$. Let v, w be in V. Show that
$\dim\ \mathrm{span}\ \{v, w\} \leq 1$ if and only if $| < v, w > | = \|v\|\|w\|$.

<u>exercise 9</u>:

Let $(V, <, >)$ be a finite-dimensional inner product space and W a subspace. Let $\{e_1, ..., e_p, ..., e_n\}$ be an orthonormal basis of V such that $\{e_1, ..., e_p\}$ is a basis of W. Show that $\{e_{p+1}, ..., e_n\}$ is a basis of $W^{\perp}$ and infer that W=$W^{\perp\perp}$.

exercise 10:
Let $(V, <, >)$ be a finite-dimensional inner product space.Let W be a subspace of V, and P the orthogonal projection on W.
(i). Show that Ker $P = \text{Im } (I - P)$.
(ii). Show that for all x in V,

$$\inf_{y \in W} \|x - y\| = \|x - Px\|.$$

exercise 11:
Let A be in $\mathbb{R}^{m \times n}$ and b in $\mathbb{R}^m$. Let P be the orthogonal projection in $\mathbb{R}^m$ on Im A.
(i). Show that the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|Ax - b\|$ achieves its minimum at some x_0 in $\mathbb{R}^n$ such that $Ax_0 = Pb$.
(ii). Show that $A^T Ax_0 = A^T b$.