<u>exercise 1</u>:
Let V, W be two vector spaces and S in $\mathcal{L}(V, W)$. Let V_1 be a subspace of V and W_1 be a subspace of W. Show that $S(V_1)$ is a subspace of W and that $S^{-1}(W_1)$ is a subspace of V.

<u>exercise 2</u>:
Let A be in $K^{n\times n}$ such that for all B be in $K^{n\times n}$, $AB = BA$. Show that A is a multiple of the identity matrix. **Hint:** In a first step use the basis matrices E_{ij} to show that A is diagonal. Alternatively, you may use elementary matrices.

<u>exercise 3</u>:
Let U, V, W be three vector spaces. Assume that U, V are finite dimensional. Let S in $\mathcal{L}(V, W)$, T in $\mathcal{L}(U, V)$. Show that

$$\dim \operatorname{Ker} ST \leq \dim \operatorname{Ker} S + \dim \operatorname{Ker} T.$$

Hint: First show that Ker $T \subset$ Ker $ST \subset U$.

<u>exercise 4</u>:
To answer the following questions, you may want to use polynomials.
(i). Find a vector space V and $T \in \mathcal{L}(V)$ such that T is injective but T is not surjective.
(ii). Find a vector space V and $S \in \mathcal{L}(V)$ such that S is surjective but S is not injective.

<u>exercise 5</u>:
Look up the definition of the trace then solve 2.3.27. Suggestion: Introduce $C = AB$ and $D = BA$. Write the i, j entries of C and of D using the entries of A and B.

<u>exercise 6</u>:
If V, W are finite-dimensional spaces over K, show that

$$\dim \mathcal{L}(V, W) = \dim V \quad \dim W,$$

by finding an explicit basis for $\mathcal{L}(V, W)$ from bases of V and W.

<u>exercise 7</u>:
Let V be a finite-dimensional space. Show that the subspace H is a hyperplane if and only if it is the kernel of a non-zero element in V'.