exercise 1:

Let V be a n-dimensional vector space over K.
(i). Let B be in $\mathcal{L}(V)$. Assume that B is diagonalizable and that W is a subspace that is stable under B. Show that the restriction of B to W is diagonalizable.

Let A be in $\mathcal{L}(V)$ such that $P(X) = (X - \lambda_1)^{m_1}...(X - \lambda_p)^{m_p}$ is the minimal polynomial of A, where $\lambda_1, ..., \lambda_p$ are distinct (look up the definition of the minimal polynomial in your textbook). Set $W_j = \text{Ker } (A - \lambda_j I)^{m_j}$, $j = 1, ..., p$. Suppose that $A = D + N$ where D is diagonalizable, N is nilpotent, and $DN = ND$.
(ii). Show that W_j is stable under A, D, and N.
Let A_j, D_j, N_j be the restrictions of A, D, and N to W_j.
(iii). Show that $D_j = \lambda_j I_{W_j}$.
(iv). Conclude that D and N are unique.

exercise 2:

Let V be a complex vector space such that $\dim V = n$ and T in $\mathcal{L}(V)$ be such that $\text{Ker } T^{n-1} \neq \text{Ker } T^{n-2}$. Show that T has at most two distinct eigenvalues.

exercise 3:

Let $A \in K^{n\times n}$ and $\|A\|_\infty = \sup\limits_{x\in K^n, \|x\|_\infty=1} \|Ax\|_\infty$. Show that $\|A\|_\infty = \max\limits_{1\le i\le n} \sum\limits_{j=1}^{n} |a_{ij}|$.

exercise 4:

Let $A \in K^{n\times n}$. Show that $\sum\limits_{k=0}^{\infty} A^k$ converges if and only if $\rho(A) < 1$.

exercise 5:

Let $A \in K^{n\times n}$ and $\| \, \|$ a vector derived norm on $K^{n\times n}$.
(i). Show that $\|A^k\| \ge \rho(A)^k$.
(ii). Fix $\epsilon > 0$. Set $B = (\rho(A) + \epsilon)^{-1} A$. Find $\lim\limits_{k\to\infty} B^k$.
(iii). Find $\lim\limits_{k\to\infty} \|A^k\|^{\frac{1}{k}}$.