<u>exercise 1</u>:
Let V be a finite-dimensional vector space over $\mathbb{R}$ and A a symmetric operator in $\mathcal{L}(V)$. Let $n = \dim V$ and $\lambda_1 < ... < \lambda_p$ be the ordered eigenvalues of A.
(i). Show that $\lambda_p = \max_{x \in V, \|x\|=1} < Ax, x >$.
(ii). Show that $\lambda_1 = \min_{x \in V, \|x\|=1} < Ax, x >$.
(iii). Let $E_p = \text{Ker } (A - \lambda_p I)$. Show that $\lambda_{p-1} = \max_{x \in E_p^{\perp}, \|x\|=1} < Ax, x >$.
(iv). If $\lambda_1 \geq 0$ find $\max_{x \in V, \|x\|=1} \|Ax\|$.

<u>exercise 2</u>:
Find a complex square matrix M such that $M^T = M$ and M is not diagonalizable.

<u>exercise 3</u>:
Let V be a finite-dimensional vector space over $\mathbb{C}$ and A a Hermitian operator in $\mathcal{L}(V)$ such that its eigenvalues are in $[0, \infty)$. Show that the eigenvalues of A are equal to the square roots of the eigenvalues of A^*A.

<u>exercise 4</u>:
Let A be in $\mathbb{R}^{m \times n}$ and b in $\mathbb{R}^m$. Let P be the orthogonal projection in $\mathbb{R}^m$ on Im A.
(i). Show that the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|Ax - b\|$ achieves its minimum at some x_0 in $\mathbb{R}^n$ such that $Ax_0 = Pb$. **Hint:** It is convenient to manipulate f^2.
(ii). Show that $A^*Ax_0 = A^*b$.

<u>exercise 5</u>:
Let V be a finite-dimensional inner product space over $\mathbb{R}$ and U a unitary operator in $\mathcal{L}(V)$. Let M be the matrix of U in an orthonormal basis of V.
(i). Show that M is normal and that any complex eigenvalue λ of M satisfies $|\lambda| = 1$.
(ii). Show that if λ is an eigenvalue of M, then $\overline{\lambda}$ is an eigenvalue of M.
(iii). Prove theorem 6.4 , section 8.6, from the textbook using complex normal operators as discussed in class. Hint: let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ be an eigenvalue of M and v an eigenvector. Use $\frac{1}{\sqrt{2}}(v + \overline{v})$ and $\frac{1}{\sqrt{2}i}(v - \overline{v})$,

<u>exercise 6</u>:
Let $e_1, ..., e_n$ be the natural basis of K^n. Let $a_0, ..., a_{n-1}$ be n scalars in K and M in $K^{n \times n}$ such that $Me_j = e_{j+1}$, if $j = 1, ..., n-1$ and

$$Me_n = -a_0 e_1 - a_1 e_2 ... - a_{n-1} e_n.$$

(i). Write down the matrix M.
(ii). Find $M^j e_1$ for $j = 0, ..., n$.

(iii). Prove that the characteristic polynomial of M is $X^n + a_{n-1}X^{n-1} + ... + a_0$.

<u>exercise 7</u>:
Let V be a vector space and $W, W_1, W_2, ..., W_p$ be subspaces. Assume that $V = W_1 \oplus W$ and $W = W_2 \oplus ... \oplus W_p$.
Show that $V = W_1 \oplus W_2 \oplus ... \oplus W_p$.

<u>exercise 8</u>:
Exercise 9.2.2.

<u>exercise 9</u>:
Exercise 9.2.5.