exercise 1:

Let $[a, b]$ be an interval in $\mathbb{R}$ and $x_0 < x_1 < ... < x_n$ be in $[a, b]$. Let $y_0, ..., y_n$ be in $\mathbb{R}$.
(i). Use exercise 6 from homework 1 to show that there is a unique polynomial P in $\mathbb{R}[X]$ with $\deg P \leq n$ such that $P(x_j) = y_j$, for $j = 0, ..., n$.
(ii). Show the same result using Vandermonde determinants.

exercise 2:

Let A be in $K^{n\times n}$. Assume that A^T is singular.
(i). Let $R_1, ..., R_n$ be the rows of A. Show that there are $a_1, ..., a_n$ that are not all zero and such that $a_1R_1 + ... + a_nR_n = 0$.
(ii). Infer that the image space of A is included in a hyperplane.
(iii). Infer that A is singular.

exercise 3:

Let V be a vector space and V_1, V_2 two subspaces such that $V = V_1 \bigoplus V_2$. Assume that $\{e_1, ..., e_m\}$ is a basis of V_1 and that $\{f_1, ..., f_n\}$ is a basis of V_2. Show that $\{e_1, ..., e_m, f_1, ..., f_n\}$ is a basis of V.

exercise 4:

Let A be in $\mathbb{R}^{n\times n}$ and define $\varphi(x, y) = x^TAy$ for x, y in $\mathbb{R}^n$.
(i). Show that φ is symmetric if and only if A is symmetric.
(ii). Assume that φ is symmetric and non-negative, that is, $\varphi(x, x) \geq 0$ for all x in $\mathbb{R}^n$. Show that φ is definite positive if and only if A is invertible.

exercise 5:

Let $(V, <, >)$ be a Euclidean vector space over $\mathbb{R}$. Let v, w be in V. Show that
$\dim\ \text{span}\ \{v, w\} \leq 1$ if and only if $| < v, w > | = \|v\|\|w\|$.

exercise 6:

Let $(V, <, >)$ be a finite-dimensional inner product space and W a subspace. Let $\{e_1, ..., e_p, ..., e_n\}$ be an orthonormal basis of V such that $\{e_1, ..., e_p\}$ is a basis of W. Show that $\{e_{p+1}, ..., e_n\}$ is a basis of $W^\perp$ and infer that W=$W^{\perp\perp}$.

exercise 7:

Let $(V, <, >)$ be a finite-dimensional inner product space.Let W be a subspace of V, and P the orthogonal projection on W.
(i). Show that Ker $P =$ Im $(I - P)$.

(ii). Show that for all x in V,

$$\inf_{y \in W} \|x - y\| = \|x - Px\|.$$

<u>exercise 8</u>:
Textbook exercise 5.3.2.

<u>exercise 9</u>:
Textbook exercise 5.5.2.