exercise 1:

Let V, W be two vector spaces and S in $\mathcal{L}(V, W)$. Let V_1 be a subspace of V and W_1 be a subspace of W. Show that $S(V_1)$ is a subspace of W and that $S^{-1}(W_1)$ is a subspace of V.

exercise 2:

Let U, V, W be three vector spaces. Assume that U, V are finite dimensional. Let S in $\mathcal{L}(V, W)$, T in $\mathcal{L}(U, V)$. Show that

$$\dim \operatorname{Ker} ST \leq \dim \operatorname{Ker} S + \dim \operatorname{Ker} T.$$

exercise 3:

To answer the following questions, you may want to use polynomials.
(i). Find a vector space V and $T \in \mathcal{L}(V)$ such that T is injective but T is not surjective.
(ii). Find a vector space V and $S \in \mathcal{L}(V)$ such that S is surjective but S is not injective.

exercise 4:

Look up the definition of the trace then solve 2.3.27. Suggestion: Introduce $C = AB$ and $D = BA$. Write the i, j entries of C and of D using the entries of A and B.

exercise 5:

2.3.35. Suggestion: prove by induction on k that $A^k e_j = 0$ for all $j = 1, .., k$.

exercise 6:

If V, W are finite-dimensional spaces over K, show that

$$\dim \mathcal{L}(V, W) = \dim V \quad \dim W,$$

by finding an explicit basis for $\mathcal{L}(V, W)$ from bases of V and W.

exercise 7:

Let V be a finite-dimensional space. Show that the subspace H is a hyperplane if and only if it is the kernel of a non-zero element in V'.

exercise 8:

Let A be in $K^{n\times n}$ such that for all B be in $K^{n\times n}$, $AB = BA$. Show that A is a multiple of the identity matrix. **Hint:** In a first step use the basis matrices E_{ij} to show that A is diagonal. Alternatively, you may use elementary matrices.