<u>exercise 1</u>:

Let V be a finite-dimensional vector space over $\mathbb{R}$ and A a symmetric operator in $\mathcal{L}(V)$. Let $n = \dim V$ and $\lambda_1 < .. < \lambda_p$ be the ordered eigenvalues of A.
(i). Show that $\lambda_p = \max\limits_{x \in V, \|x\|=1} < Ax, x >$.
(ii). Show that $\lambda_1 = \min\limits_{x \in V, \|x\|=1} < Ax, x >$.
(iii). Let $E_p = \text{Ker }(A - \lambda_p I)$. Show that $\lambda_{p-1} = \max\limits_{x \in E_p^\perp, \|x\|=1} < Ax, x >$.
(iv). If $\lambda_1 \geq 0$ find $\max\limits_{x \in V, \|x\|=1} \|Ax\|$.

<u>exercise 2</u>:

Find a complex square matrix M such that $M^T = M$ and M is not diagonalizable.

<u>exercise 3</u>:

Let V be a finite-dimensional vector space over $\mathbb{C}$ and U a unitary operator in $\mathcal{L}(V)$.
(i). Show that U is normal and that any eigenvalue λ of U satisfies $|\lambda| = 1$.
(ii). Let M be the matrix of U in an orthonormal basis of V. Assume that the entries of M are all real. Let λ be an eigenvalue of U. Show that $\overline{\lambda}$ is an eigenvalue of U.
(iii). Prove theorem 6.4 , section 8.6, from the textbook using complex normal operators as discussed in class.

<u>exercise 4</u>:

Let V be a finite-dimensional vector space over $\mathbb{C}$ and A a Hermitian operator in $\mathcal{L}(V)$ such that its eigenvalues are in $[0, \infty)$. Show that the eigenvalues of A are equal to the square roots of the eigenvalues of A^*A.

<u>exercise 5</u>:

Let A be in $\mathbb{R}^{m \times n}$ and b in $\mathbb{R}^m$. Let P be the orthogonal projection in $\mathbb{R}^m$ on Im A.
(i). Show that the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|Ax - b\|$ achieves its minimum at some x_0 in $\mathbb{R}^n$ such that $Ax_0 = Pb$.
(ii). Show that $A^*Ax_0 = A^*b$.