exercise 1:

*** this optional problem is the continuation of hw 2 ex 1***

(vii). We proved that for any integer $k$ greater than 2,

$$\frac{1}{2}(\ln(k-1)+\ln k) \leq \int_{k-1}^{k} \ln x\, dx \leq \ln(k-\frac{1}{2})$$

Show that the series

$$\sum_{k=2}^{\infty} \int_{k-1}^{k} \ln x\, dx - \frac{1}{2}(\ln(k-1)+\ln k)$$

converges. **Hint:** It suffices to show that

$$\sum_{k=2}^{\infty} \ln(k-\frac{1}{2}) - \frac{1}{2}(\ln(k-1)+\ln k)$$

converges.

(viii). Infer that the sequence $\dfrac{n!}{n^{n+\frac{1}{2}}e^{-n}}$ converges to a limit $C$. Determine $C$.

exercise 2:

Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n \cos(nx)}{x^2+n^3+1}$ defines a differentiable function $g(x)$ for $x$ in $[-1,1]$.

exercise 3:

Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{x^2+n^2+1}$ defines a differentiable function $g(x)$ for $x$ in $(-1,1)$.

exercise 4:

(i). Show that the series of functions $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$ converges pointwise for $x$ in $(-1,1)$ and write the resulting sum in closed form.

(ii). Fix $A$ in $(0,1)$. Show that $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$ is uniformly convergent in $[-A,A]$.

(iii). Show that $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, for all $x$ in $(-1,1)$. Find $a_n$.

exercise 5:

Let $f:[0,1]\to\mathbb{R}$ be a Riemann integrable function. Let $M>0$ be such that $|f(x)|\leq M$, for all $x$ in $[0,1]$.

(i). Show that $\displaystyle\lim_{n\to\infty}\int_0^1 x^n f(x)dx = 0$.

(ii). Let $\alpha$ be a real number such that $0 < \alpha < 1$. Show that $\lim\limits_{n\to\infty} \displaystyle\int_0^{1-\alpha} nx^n f(x)dx = 0$.

(iii). Assume $f$ is continuous at 1. Show that $\lim\limits_{n\to\infty} \displaystyle\int_0^1 nx^n f(x)dx = f(1)$.

**Hint:** First prove it if $f$ is a constant function, then use answer from question (ii). by splitting the interval of integration in two.