exercise 1:

*** this optional problem is the continuation of hw 2 ex 1***

(viii). We proved that for any integer k greater than 2,

$$\frac{1}{2}(\ln(k-1)+\ln k) \leq \int_{k-1}^{k} \ln x \, dx \leq \ln(k-\frac{1}{2})$$

Show that the series

$$\sum_{k=2}^{\infty} \int_{k-1}^{k} \ln x \, dx - \frac{1}{2}(\ln(k-1)+\ln k)$$

converges. **Hint:** It suffices to show that

$$\sum_{k=2}^{\infty} \ln(k-\frac{1}{2}) - \frac{1}{2}(\ln(k-1)+\ln k)$$

converges.

(ix). Infer that the sequence $\dfrac{n!}{n^{n+\frac{1}{2}}e^{-n}}$ converges to a limit C. Determine C.

exercise 2:

Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n \cos(nx)}{x^2+n^3+1}$ defines a differentiable function $g(x)$ for x in $[-1,1]$.

exercise 3:

Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{x^2+n^2+1}$ defines a differentiable function $g(x)$ for x in $(-1,1)$.

exercise 4:

Set $f_n : [-1,1] \to \mathbb{R}$, $f_n(x) = (x^2 + \frac{1}{n})^{\frac{1}{2}}$.

(i). Show that f_n' is pointwise convergent to a function g.

(ii). Find a lower bound for $\sup\limits_{x\in[-1,1]} |f_n'(x) - g(x)|$ that shows that this convergence is not uniform.

exercise 5:

Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function. Let $M > 0$ be such that $|f(x)| \leq M$, for all x in $[0,1]$.

(i). Show that $\lim\limits_{n\to\infty} \displaystyle\int_0^1 x^n f(x)dx = 0$.

(ii). Let α be a real number such that $0 < \alpha < 1$. Show that $\lim\limits_{n\to\infty} \displaystyle\int_0^{\alpha} nx^n f(x)dx = 0$.

(iii). Assume f is continuous at 1. Show that $\lim_{n\to\infty}\int_0^1 nx^n f(x)dx = f(1)$.
Hint: First prove it if f is a constant function, then use answer from question (ii). by splitting the interval of integration in two.

exercise 6:
8.3.E. **Hint:** Set $g(x,t) = \frac{\sin xt}{t}$ if $t \neq 0$, $g(x,0) = x$.

exercise 7:
8.3.H. **Hint:** Set $G(x,y) = \int_0^y f(x-t)d(t)dt$ for $0 \leq x \leq 1, 0 \leq y \leq x$.