Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

LOCAL MINIMALITY AND CRACK PREDICTION IN QUASI-STATIC GRIFFITH FRACTURE EVOLUTION

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Abstract. The mathematical analysis developed for energy minimizing fracture evolutions has been difficult to extend to locally minimizing evolutions. The reasons for this difficulty are not obvious, and our goal in this paper is to describe in some detail what precisely the issues are and why the previous analysis in fact cannot be extended to the most natural models based on local minimality. We also indicate how the previous methods can be modified for the analysis of models based on a recent definition of stability that is a bit stronger than local minimality.

1. **Introduction.**

Inspired by Griffith's criterion for crack growth ([14]), there has been substantial mathematical progress on global energy-minimization models for crack prediction in which the energy of a crack is proportional to its surface area [13, 9, 12, 7, 8]. However, Griffith's criterion is explicitly local, and while formulating local energy-minimization models for crack prediction can be done, the analysis that has been developed for the global minimization setting does not extend in a straightforward way to local minimization.

More precisely, the issue is combining a certain notion of *unilateral* with global and local minimality. This unilateral minimality comes from the fact that the models treat fracture as irreversible, meaning that while there is an energy cost for the creation of cracks, modeled as discontinuities of the displacement, there is no energy reduction if at some later time a discontinuity disappears. Hence, the only minimality property that a displacement u can have relative to other displacements v is

$$E_{el}(u) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus C), \tag{1}$$

where u and v are in $SBV(\Omega)$, E_{el} is stored elastic energy, S_v is the discontinuity set, or jump set, of v, $\mathcal{H}^{N-1}$ is the $N-1$-dimensional Hausdorff measure, C is the crack set, and $S_u \subset C$ (see [4] for definitions of SBV and jump sets). We note that for simplicity in highlighting some of the issues that we discuss below, we will sometimes suppose that $S_u = C$. The unilateral minimality is then

$$E_{el}(u) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus S_u).$$

Of course, the issue of global vs. local minimality then concerns the class of v for which the above inequality holds – for global minimality, it holds for all v

2000 *Mathematics Subject Classification.* Primary: 74R10; Secondary: 49Q20 74G65 74H55.
Key words and phrases. Fracture mechanics, Quasi-static evolution, rate independent.

(constrained, e.g., to the appropriate Dirichlet condition), and for local minimality, it holds for all v close enough to u, constrained as above.

For quasi-static evolutions, we seek a pair $(u(t), C(t))$, $t \in [0,T]$, such that $u(t)$ satisfies a given Dirichlet boundary condition $f(t)$, and such that, among other things, the above unilateral minimality (1) holds for $u(t)$, with crack set $C(t)$. Following [2] and [13], the usual strategy is to consider (nested) sets of discrete times $\{t_i^n\} \subset \{t_i^{n+1}\} \subset [0,T]$ such that $t_0^n = 0, t_n^n = T$, and $\max(t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$. One defines $u_n(t_i^n)$ recursively by minimizing

$$v \mapsto E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus C_n(t_{i-1}^n)),$$

over $v \in SBV(\Omega)$, $v = f(t_i^n)$ on $\partial\Omega$, where $C_n(t_{i-1}^n) := \bigcup_{j<i} S_{u_n(t_j^n)}$. The fact that these minimizers exist follows from [1]. Then, the basic idea is to consider the countable dense set $D \subset [0,T]$ that is the union of all the discrete times, and then for $t \in D$, take the limit of $u_n(t)$ as $n \to \infty$, again using [1]. This gives $u(t)$ for $t \in D$, and then C is defined for all t by

$$C(t) := \bigcup_{\substack{\tau \le t \\ \tau \in D}} S_{u(\tau)}.$$

u is extended to $[0,T]$, either by taking limits, or by minimizing the elastic energy over $v \in SBV$ with $S_v \subset C(t)$ (and the correct Dirichlet condition).

It then remains to show that for every t, $u(t)$ has the desired minimality (1), with crack set $C(t)$. This has been the main difficulty and indeed the main reason for assuming the crack path a priori, or restrictions on the crack path. In a nutshell, the problem is to conclude that if $u_n \overset{SBV}{\rightharpoonup} u$ (see definition 1.1 below) and the u_n are unilateral minimizers, then so is u. The main point of [12] was to prove this minimality of limits, which is the essence of the Jump Transfer theorem, described below.

However, while this method handles global minimality, addressing local minimality is much more subtle. Indeed, the conclusion that u is a unilateral *local* minimizer if u_n are and $u_n \overset{SBV}{\rightharpoonup} u$ is false (see section 3). Of course, this is only an issue when trying to predict the crack path, and there are a number of interesting papers that study crack propagation with an a priori path (e.g., [15, 18]) or a weakening of local minimality ([10]).

The main point of this paper is to explain this, as well as precisely how the Jump Transfer argument fails. In addition, we also describe a new stability condition defined in [16], and how Jump Transfer can be strengthened (see Path Transfer, section 5 below) to prove that u is stable if u_n are and $u_n \overset{SBV}{\rightharpoonup} u$.

We now define the space SBV_p and the main convergence, SBV convergence, that is central in the analysis of these fracture problems (see [1]).

Definition 1.1. We set $SBV_p(\Omega) := \{v \in SBV(\Omega) : \nabla v \in L^p(\Omega)\}$. We say that $u_n \overset{SBV}{\rightharpoonup} u$, or u_n converges to u in the sense of SBV convergence, if $u_n, u \in SBV(\Omega)$ and

$$\begin{cases} \nabla u_n \rightharpoonup \nabla u \text{ in } L^1(\Omega); \\ [u_n]\nu_n \mathcal{H}^{N-1} \lfloor S_{u_n} \overset{*}{\rightharpoonup} [u]\nu \mathcal{H}^{N-1} \lfloor S_u \text{ as measures;} \\ u_n \to u \text{ in } L^1(\Omega); \text{ and} \\ u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(\Omega). \end{cases}$$

We remark that while the space SBD (see [3, 5]) is natural for linearized elasticity, there is currently no Jump Transfer theorem for that space, essentially because there is no (known) analog of the coarea formula. Consequently, at this time, there are no crack path predicting results for quasi-static fracture in SBD.

2. **The mathematical method: Jump Transfer.** As explained in the introduction, the unilateral minimality that we seek is

$$E_{el}(u) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus S_u)$$

for all $v \in SBV(\Omega)$, where u is the limit in SBV of a sequence u_n having this unilateral minimality (with $\mathcal{H}^{N-1}(S_v \setminus S_{u_n})$ instead of $\mathcal{H}^{N-1}(S_v \setminus S_u)$, of course). Generally this elastic energy is of the form $E_{el}(u) = \int_\Omega W(\nabla u)dx$ and lower semicontinuous with respect to SBV convergence. The standard approach is to take a sequence of u_n that have this property, by construction, take a subsequence converging to some $u \in SBV(\Omega)$, and seek to show that therefore u satisfies this minimality. Jump Transfer is the tool for showing this ([12]).

The argument is the following. We assume the simplified unilateral minimality of u_n,

$$E_{el}(u_n) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus S_{u_n})$$

and seek to show

$$E_{el}(u) \leq E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus S_u)$$

for the same class of v (e.g., with a specified Dirichlet condition). As mentioned, we have $E_{el}(u) \leq \liminf_{n\to\infty} E_{el}(u_n)$, so the issue (seemingly) is only whether

$$\lim_{n\to\infty} \mathcal{H}^{N-1}(S_v \setminus S_{u_n}) \leq \mathcal{H}^{N-1}(S_v \setminus S_u).$$

In fact, in general this is false, as, e.g., it might be that $S_v \subset S_u$, so the right hand side above is zero, while the left hand side is positive. The essence of Jump Transfer is to alter any such v, creating v_n, that makes a small change in E_{el}, while "transferring" the part of S_v that is inside S_u into S_{u_n}. We then get v_n such that $E_{el}(v_n) \to E_{el}(v)$ while

$$\lim_{n\to\infty} \mathcal{H}^{N-1}(S_{v_n} \setminus S_{u_n}) \leq \mathcal{H}^{N-1}(S_v \setminus S_u),$$

which gives the unilateral minimality of u.

We now state and sketch the proof of Jump Transfer. We give the theorem as originally formulated in [12], and note that the inclusion $\overline{\Omega} \subset \Omega'$ is just a device for combining Dirichlet conditions with fracture. The issue is that fracture might occur along the boundary of the domain, so that the Dirichlet data might not be met. A simple way for handling this is to consider a larger domain $\Omega' \supset\supset \Omega$, so that the Dirichlet condition is met on $\Omega' \setminus \overline{\Omega}$, and fracture can occur only in $\overline{\Omega}$.

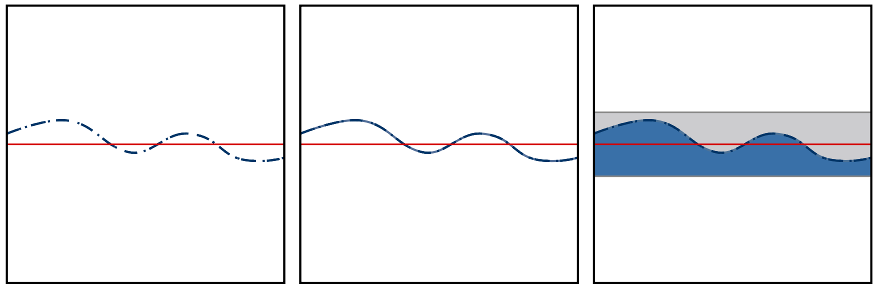
Theorem 2.1 (Jump Transfer). *Let $\overline{\Omega} \subset \Omega'$, with $\partial\Omega$ Lipschitz, and let $\{u_n\} \subset SBV(\Omega')$ be such that*

- $S_{u_n} \subset \overline{\Omega}$;
- $|\nabla u_n|$ *weakly converges in* $L^1(\Omega')$; *and*
- $u_n \to u$ *in* $L^1(\Omega')$,

where $u \in BV(\Omega')$ with $\mathcal{H}^{N-1}(S_u) < \infty$. Then, for every $v \in SBV_p(\Omega')$, $1 \leq p < \infty$, with $\mathcal{H}^{N-1}(S_v) < \infty$, there exists a sequence $\{v_n\} \subset SBV_p(\Omega')$ with $v_n = v$ on $\Omega' \setminus \overline{\Omega}$ such that

i) $v_n \to v$ *strongly in* $L^1(\Omega')$;

Figure 1: Jump Transfer
Left: S_u (center line) and S_{u_n}; **Center**: S_u and $\partial_* E^n_t$; **Right**: defining v_n



ii) $\nabla v_n \to \nabla v$ *strongly in* $L^p(\Omega')$*; and*

iii) $\mathcal{H}^{N-1}\Big([S_{v_n} \setminus S_{u_n}] \setminus [S_v \setminus S_u]\Big) \to 0.$

In order to explain the limitations in applying this to local minimization, we first briefly describe the proof. The idea is to localize to cubes Q covering S_u, in which $Q \cap S_u$ is very close to a hyperplane (the center line in the left panel of Figure 1). Then, due to $u_n \overset{SBV}{\rightharpoonup} u$, we have that for n large, S_{u_n} (the broken curve in the left panel) is close to S_u in Q.

The goal is then, for a given $v \in SBV$ as in the theorem, to create $v_n \in SBV$ by redefining v in the union of these cubes such that $S_v \cap S_u$ is moved, or transferred, to S_{u_n}. Now, if S_{u_n}, which might have gaps as illustrated in the left panel, could be completed, as in the center panel, then it is a relatively simple matter to redefine v, creating v_n, so that each v_n has the part of it's jump set in S_u moved to S_{u_n} – either, as was done in [12], by reflecting v from the bottom white region in the right panel of Figure 1 into the dark region "below" S_{u_n}, and reflecting v from the upper white region into the the light region "above" S_{u_n}. Of course, this is not the only way to redefine v in these regions, and depending on the problem, other ways might be preferable. For example, in [8], dilations were used in order to maintain a determinant constraint on ∇v. The key is just to extend v into the dark region from below, and into the light region from above, so that the jump set of v that was in S_u is now inside S_{u_n}.

The main difficulty, however, is that S_{u_n} might have gaps, as in the left panel. The central idea in Jump Transfer is that there exists a $t \in \mathbb{R}$ such that, setting $E^n_t := \{x : u_n(x) > t\}$, we have $\mathcal{H}^{N-1}(Q \cap \partial_* E^n_t \setminus S_{u_n})$ is small, where ∂_* denotes the measure theoretic boundary (see, e.g., [11]). The above construction of v_n is then performed, where the region "below" S_{u_n} is simply $E^n_t \cap Q$ (assuming the orientation of Q is such that the larger value of u is "below" S_u). The effect of this construction on the total energy of v includes the addition of the (arbitrarily small, for n large) increments $\mathcal{H}^{N-1}(Q \cap \partial_* E^n_t \setminus S_{u_n})$, as well as (also small) new discontinuity sets created on the boundary of each cube, since S_u and S_{u_n} do not necessarily coincide on these boundaries ∂Q.

3. **Problems with local minimization.** If one seeks a model based on local minimality rather than global, there is immediately the issue of what to do at the discrete-time stage, to find $u_n(t^n_{i+1})$ from $u_n(t^n_i)$, instead of globally minimizing the energy (unilaterally). Whatever is done, it must be such that if, for example, $u_n(t_i)$ is at the bottom of a local energy well for the energy at time t_{i+1}, then we ought to

have $u_n(t_{i+1}) = u_n(t_i)$. Furthermore, this must be the case even if there exists some v with lower energy, otherwise it would just be an algorithm for globally minimizing evolutions. Of course, this principle is the same if $u_n(t_i)$ is in a local energy well at time t_{i+1}, even if it is not at the exact bottom – at time t_{i+1}, it should just move to the bottom of that well, whether or not somewhere there is a lower energy state v.

The question then is, what kind of minimization should be performed at the discrete-time level that will produce evolutions with this property? One natural idea is to consider gradient flows for the energy corresponding to time t_{i+1}, with initial condition $u_n(t_i)$, and take the long-time limit (see [10]). More generally, one could consider all states v that are reachable by continuous paths starting at $u_n(t_i)$, with the constraint, of course, that the energy does not increase along the path, so that in particular, a state will be stuck in its local energy well. Specifically, we can define

Definition 3.1 (Slides)**.** We say that a continuous map $\phi\colon[0,1]\to SBV$ is a *slide* for $u \in SBV$ if $\phi(0) = u$, $E_\phi(\tau_2) \le E_\phi(\tau_1)$ for all $\tau_1 < \tau_2$, and $E_\phi(0) > E_\phi(1)$.

Here,

$$E_\phi(\tau) := E_{el}(\phi(\tau)) + \mathcal{H}^{N-1}\Big(C_\phi(\tau)\Big)$$

and

$$C_\phi(\tau) := \bigcup_{\substack{s\in\mathbb{Q}\\ s\le\tau}} S_{\phi(s)}.$$

In fact, it follows from lemma 6.6 of [17] (see lemma 2.2 of [16]) that:

Lemma 3.2. *For any slide ϕ, C_ϕ satisfies the following properties:*

1. *C_ϕ is increasing ($C_\phi(\tau_1) \subset C_\phi(\tau_2)$ if $\tau_1 < \tau_2$)*
2. *$S_{\phi(\tau)} \subset C_\phi(\tau)$ for all τ*
3. *C_ϕ is the smallest (in the sense of inclusion) function satisfying the previous two conditions.*

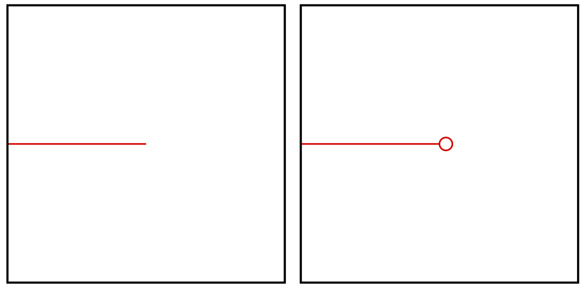
These slides could then be used to modify global minimization as follows. At each discrete time t^n_{i+1}, we take the discrete solution at the prior time, $u_n(t^n_i)$ and modify it to obtain the correct Dirichlet data at time t^n_{i+1}, to get $v_n(t^n_{i+1})$. For example, this can be done by finding $w \in H^1(\Omega)$ satisfying $\Delta w = 0$ and $w = f(t^n_{i+1}) - f(t^n_i)$ on $\partial\Omega$, and then set $v_n(t^n_{i+1}) := u_n(t^n_i) + w$. Then, we minimize

$$v \mapsto E_{el}(v) + \mathcal{H}^{N-1}(S_v \setminus C_n(t^n_i))$$

over all v that are *accessible* from $v_n(t^n_{i+1})$, meaning that there exists a slide starting at $v_n(t^n_{i+1})$ that ends at v. Note that in particular, if we consider a gradient flow (really, a minimizing movement (see, e.g., [2])) beginning at $v_n(t^n_{i+1})$, and the long time limit is v, then v is accessible. This would then be iterated in i, and each $u_n(t^n_{i+1})$ would presumably be a unilateral local minimizer. The limit $n \to \infty$ would then be taken, as described above, in order to obtain $u(t)$.

It turns out that there are two difficulties with implementing this approach, and one of them is fatal. The first is that Jump Transfer cannot work with these slides – in particular, as described above, the filling-in of S_{u_n} with $\partial_* E^n_t$ will in general add some arbitrarily small amount to the energy, of size at least $\mathcal{H}^{N-1}(Q \cap \partial_* E^n_t \setminus S_{u_n})$ in each cube Q used to cover S_u. Hence, for a suitable path φ starting at $u(t)$, along which the total energy does not increase, if $u_n(t^n_i) \overset{SBV}{\rightharpoonup} u(t)$, when we transfer this

Figure 2: Local minimizers can converge to non-local minimizers



path to $u_n(t_i^n)$, it is no longer a slide. The main consequence is that if $u(t)$ is not a local minimizer, and so there exists a slide starting at $u(t)$, this slide cannot be transferred to $u_n(t_i^n)$ in a way that keeps it a slide. Therefore, it is consistent with Jump Transfer that

$$u_n \stackrel{SBV}{\rightharpoonup} u, \; u_n \text{ are all (strict) local minimizers, but } u \text{ is not.} \tag{2}$$

In fact, this is not just a failure of this method – it turns out that it is possible to have (2), as is easy to see with a simple counterexample: suppose we have $u \in SBV$ with jump set S_u that has a single tip, e.g., S_u is a line segment with one end in the boundary of Ω and the other end (the crack tip) in Ω (see the left panel in Figure 2). Suppose further that u is in elastic equilibrium, but it is not a unilateral local minimizer, i.e., the crack can grow continuously from its tip such that the total energy is strictly decreasing. If we add a circle of radius $1/n$ at the crack tip of u (placing it so that the tip sits on the circle, not at the center, see the right panel of Figure 2), and set u_n to be the corresponding elastic equilibrium, then, since this new crack set has no tip, ∇u_n has no singularity. It is then an easy consequence that these u_n are (strict) local minimizers (see, e.g., [6]), and also that $u_n \to u$. So, unilateral local minimality simply is not maintained under weak SBV limits.

4. **Strengthening local minimality: a stability condition.** One thing that is clear from the above problems with local minimality is that, if we want a stability criterion related to the existence of slides, and that is maintained under weak SBV convergence, i.e.,

$$u_n \stackrel{SBV}{\rightharpoonup} u \text{ and the } u_n \text{ are stable } \Rightarrow u \text{ is stable,}$$

we cannot insist that the energy does not increase at all along the continuous paths to lower energy states. Fortunately, once we loosen this "no increase" condition an arbitrarily small amount, everything goes through. Precisely, our new definition ([16]) is:

Definition 4.1 (Epsilon-Slides)**.** We say that a continuous map $\phi : [0,1] \to SBV$ is an *ε-slide* for $u \in SBV$ if $\phi(0) = u$,

$$\sup_{\tau_1 < \tau_2} \big[E_\phi(\tau_2) - E_\phi(\tau_1) \big] < \varepsilon, \tag{3}$$

and $E_\phi(0) > E_\phi(1)$. A continuous map $\phi : [0,1] \to SBV$ is an *$\bar{\varepsilon}$-slide* for $u \in SBV$ if $\phi(0) = u$,

$$\sup_{\tau_1 < \tau_2} \big[E_\phi(\tau_2) - E_\phi(\tau_1) \big] \le \varepsilon, \tag{4}$$

and $E_\phi(0) > E_\phi(1)$.

Of course, Lemma 3.2 holds for ε and $\bar{\varepsilon}$-slides as well as for slides.

Definition 4.2 (Epsilon Stability)**.** $u \in SBV$ is *ε-stable* if it has no ε-slides, and *$\bar{\varepsilon}$-stable* if it has no $\bar{\varepsilon}$-slides.

As we explain in the next section, one can then prove existence of the following evolutions, which in particular have the property that at each time, they are unilateral local minimizers, but are not necessarily global minimizers.

Definition 4.3 (Epsilon-Stable Fracture)**.** (u, C) is an *ε-stable fracture* if

1. $t \mapsto C(t)$ is monotonic
2. $u(t) = g(t)$ on $\partial\Omega \setminus C(t)$ for all $t \in [0, T]$
3. $u(t)$ is ε-stable with respect to $C(t)$ for every $t \in [0, T]$
4. the limits
$$u(t)^- := \lim_{s \to t^-} u(s),\ C(t)^- := \bigcup_{s<t} C(s),$$
$$u(t)^+ := \lim_{s \to t^+} u(s),\ C(t)^+ := \bigcap_{s>t} C(s)$$
exist, and if $(u(t), C(t))^- \neq (u(t), C(t))^+$, then there exists an $\bar{\varepsilon}$-slide with respect to $C(t)^-$ from $u^-(t)$ to $u^+(t)$. Furthermore, $E(u^+(t), C^+(t)) \leq E(v, C_\phi(1))$ for every v that is ε-accessible (with ε-slide ϕ) from $u^-(t)$ with respect to $C(t)^-$.
5. for every $t_1 < t_2$,
$$E(u(t_2), C(t_2)) - E(u(t_1), C(t_1)) \leq \int_{t_1}^{t_2} \int_\Omega \nabla u \cdot \nabla \dot{g} dx dt.$$

In the above, E denotes the total energy, i.e., $E(u, C) := E_{el}(u) + \mathcal{H}^{N-1}(C)$, and for simplicity the elastic energy density is assumed to be $|\nabla u|^2$. We should also note that given any initial data u_0 that is ε-stable, there exists an evolution as above with $u(0) = u_0$.

5. **Path Transfer.** One can then prove (see Remark 4.4 in [16]) the following stronger version of Jump Transfer for paths:

Theorem 5.1 (Path Transfer)**.** *Let $\overline{\Omega} \subset \Omega'$, with $\partial\Omega$ Lipschitz, and let $\{u_n\} \subset SBV(\Omega')$ be such that*

- $S_{u_n} \subset \overline{\Omega}$*;*
- $|\nabla u_n|$ *weakly converges in* $L^1(\Omega')$*; and*
- $u_n \to u$ *in* $L^1(\Omega')$*,*

where $u \in BV(\Omega')$ with $\mathcal{H}^{N-1}(S_u) < \infty$. Then, for every path
$$\varphi : [0, 1] \to SBV_p(\Omega'),$$
$1 \leq p < \infty$, with $\mathcal{H}^{N-1}(C_\varphi(1)) < \infty$, there exists a sequence
$$\varphi_n : [0, 1] \to SBV_p(\Omega')$$
with $\varphi_n(\tau) = \varphi(\tau)$ on $\Omega' \setminus \overline{\Omega}$ $\forall \tau \in [0, 1]$, such that

i) $\varphi_n(\tau) \to \varphi(\tau)$ *strongly in* $L^1(\Omega')$ *uniformly in* τ*;*
ii) $\nabla\varphi_n(\tau) \to \nabla\varphi(\tau)$ *strongly in* $L^p(\Omega')$ *uniformly in* τ*;*
iii) $\mathcal{H}^{N-1}\Big([S_{\varphi_n(\tau)} \setminus S_{u_n}] \setminus [S_{\varphi(\tau)} \setminus S_u]\Big) \to 0$*;*
iv) $\mathcal{H}^{N-1}(C_{\varphi_n}(1) \cap T_n \setminus S_{u_n}) \to 0$*,*

where $\varphi_n = \varphi$ *outside* T_n*, and* $|T_n| \to 0$*.*

The proof is mostly based on the observation that, since $C_\varphi(1) \setminus S_u$ has density zero $\mathcal{H}^{N-1}$-a.e. in S_u, the cubes Q can be chosen such that $\mathcal{H}^{N-1}(C_{\varphi_n}(1) \cap T_n \setminus S_{u_n})$ is arbitrarily small, where T_n is the union of these cubes.

The point of the theorem is that we can construct φ_n such that, for the elastic energy density $|\nabla u|^2$,

$$\int_\Omega |\nabla \varphi_n(\tau)|^2 dx \leq \int_\Omega |\nabla \varphi(\tau)|^2 dx + O(n),$$

where $O(n)$ is uniform in τ, and similarly

$$\mathcal{H}^{N-1}(C_{\varphi_n}(\tau)) \leq \mathcal{H}^{N-1}(C_\varphi(\tau)) + O(n)$$

uniformly in τ. Hence, if u has an ε-slide φ (and it follows from the definition that it is also an $\varepsilon - \delta$-slide for some $\delta > 0$), then for n large enough that the term $O(n) < \delta$, the φ_n are ε-slides for u_n. Therefore, in contrast to (2), we have:

$$u_n \overset{SBV}{\rightharpoonup} u \text{ and the } u_n \text{ are } \varepsilon\text{-stable } \Rightarrow u \text{ is } \varepsilon\text{-stable.}$$

Finally, we note that ε-stability implies unilateral local minimality (see lemma 4.7 in [16]), and strict unilateral local minimizers are ε-stable for ε sufficiently small.

Acknowledgments. This material is based on work supported by the National Science Foundation under Grant Nos. DMS-0807825 and DMS-1009653. Thanks also to Casey Richardson for supplying the figures.

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Received xxxx 20xx; revised xxxx 20xx.

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