WORCESTER POLYTECHNIC INSTITUTE

MECHANICAL ENGINEERING DEPARTMENT

Engineering Experimentation
ME-3901, D'2012

Lecture 11

18 April 2012

General information

Office hours

Instructors: Cosme Furlong

Office: HL-151

Everyday:

9:00 to 9:50 am

Christopher Scarpino

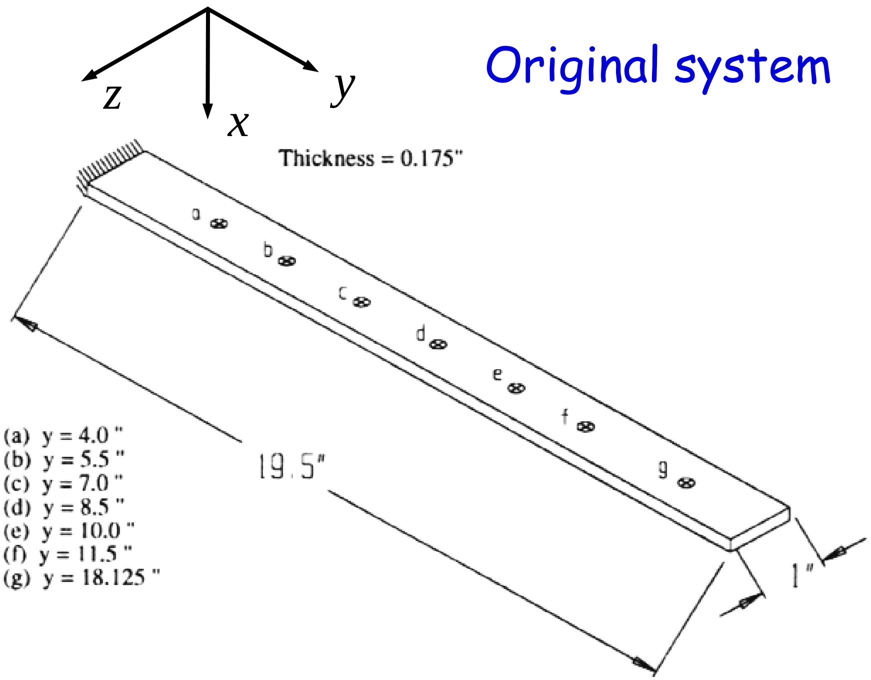
Office: HL-153

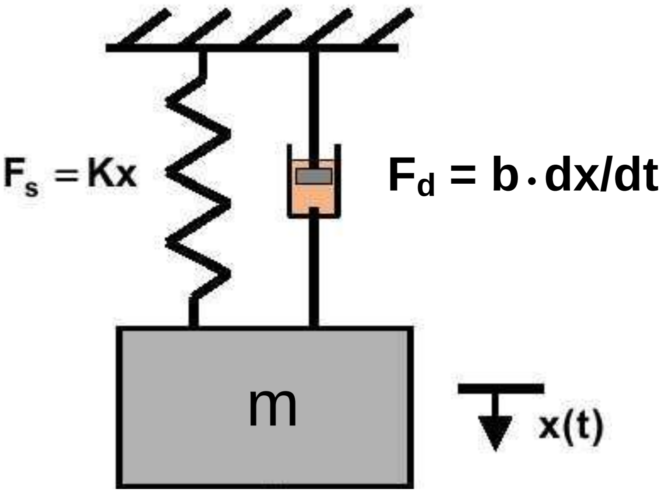
During laboratory sessions

Teaching Assistants: During laboratory sessions

Equivalent systems

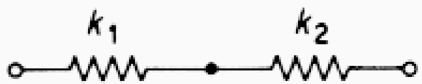
Original system → Equivalent system



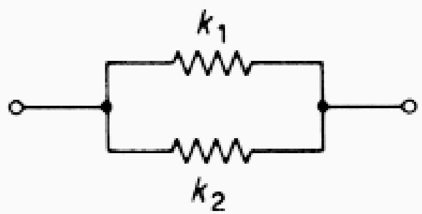


Equivalent systems

Table of Spring Stiffness



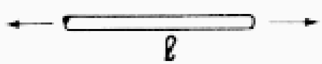
$$k = \frac{1}{1/k_1 + 1/k_2}$$



$$k = k_1 + k_2$$

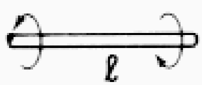
$$k = \frac{EI}{l}$$

I = moment of inertia of cross-sectional area
l = total length



$$k = \frac{EA}{l}$$

A = cross-sectional area



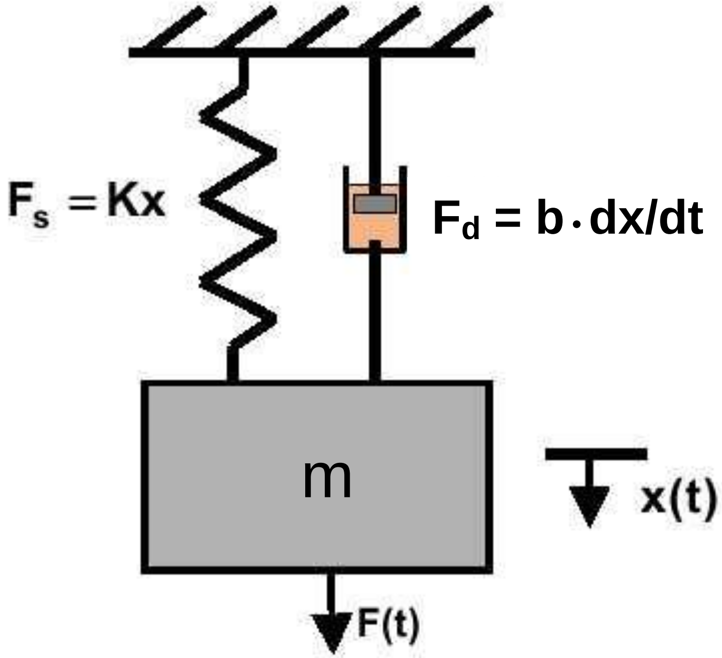
$$k = \frac{GJ}{l}$$

J = torsion constant of cross section

$$k = \frac{Gd^4}{64nR^3}$$

n = number of turns

Analysis of a single degree of freedom system



Governing equation: $$m\frac{d^2x}{dt^2} = \sum_i F_i$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F(t)$$

External force

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

Consider governing equation: $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Second order differential equation with *constant coefficients*.

Governing equation can be written as: $$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

Possible solution has the form: $$x(t) = e^{m_i t}$$

So the **characteristic equation** is: $$m_i^2 + 2\lambda m_i + \omega^2 = 0$$

Roots of characteristic equation:

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}$$

$$m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$$\lambda^2 - \omega^2 = 0$$ Critically damped system

$$\lambda^2 - \omega^2 > 0$$ Over-damped system

$$\lambda^2 - \omega^2 < 0$$ Under-damped system

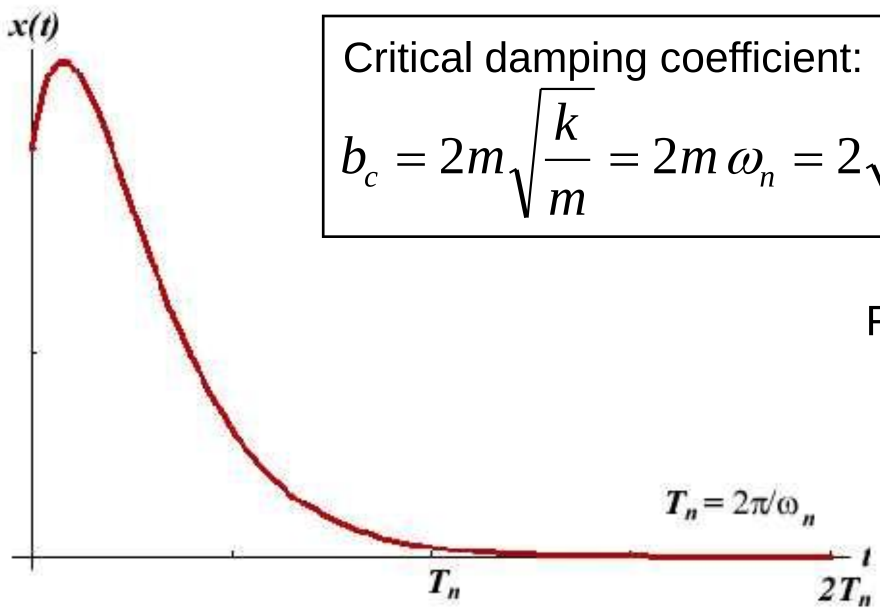
Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$$\lambda^2 - \omega^2 = 0$$ → Critically damped system

Solution to the governing differential equation:

$$x(t) = C_1 e^{m_1 t} + C_2 t e^{m_1 t}$$



Critical damping coefficient:

$$b_c = 2m\sqrt{\frac{k}{m}} = 2m\,\omega_n = 2\sqrt{k\,m}$$

Fundamental frequency:

$$\omega = \omega_n = \sqrt{\frac{k}{m}}$$

Critical damping factor b_c is the minimum damping that results in non-periodic motions

Analysis of a single degree of freedom system

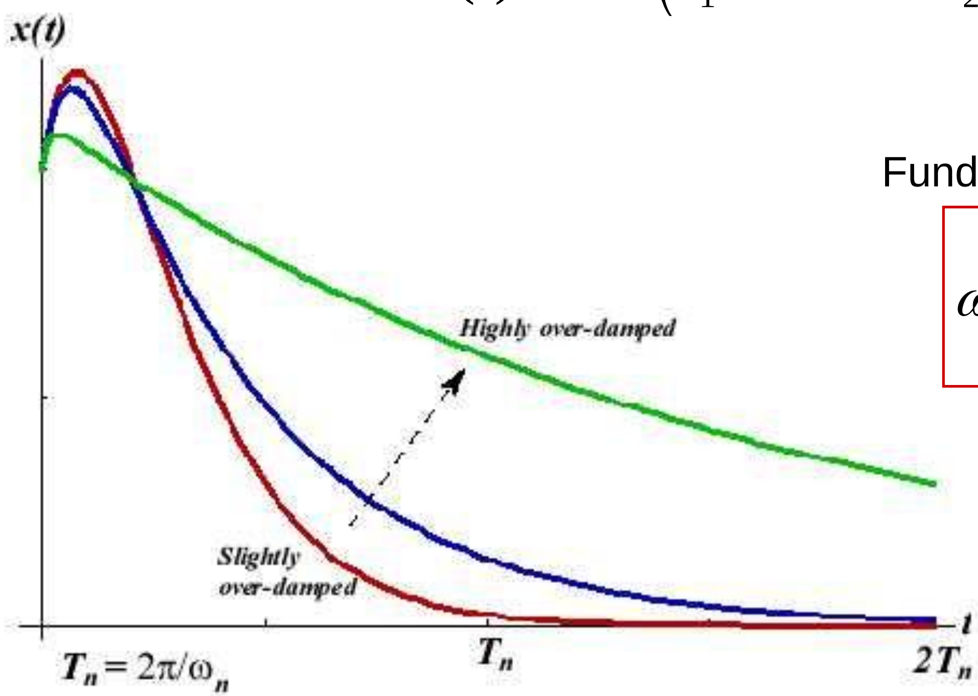
First case: $F(t) = 0$ - Free vibrations

$$\lambda^2 - \omega^2 > 0$$ → Over-damped system

Solution to the governing differential equation:

$$x(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

$$x(t) = e^{-\lambda t}\left(C_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + C_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$



Fundamental frequency:

$$\omega = \omega_n = \sqrt{\frac{k}{m}}$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

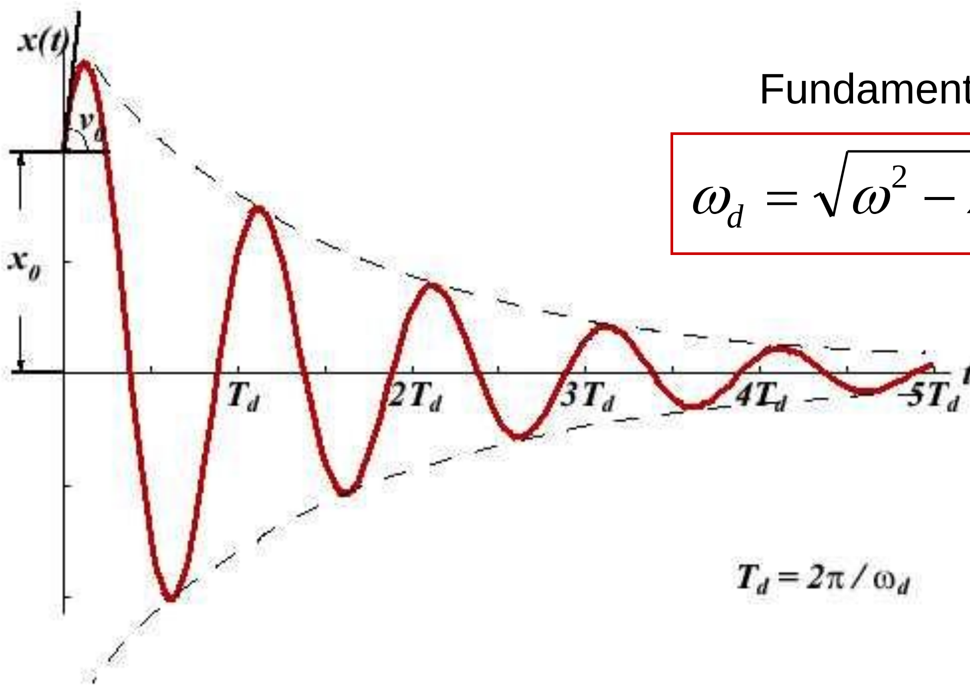
$$\lambda^2 - \omega^2 < 0$$ → Under-damped system

Solution to the governing differential equation:

$$x(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

(m_1 and m_2 are complex numbers, why?)

$$x(t) = e^{-\lambda t}[C_1 \cos(\sqrt{\omega^2 - \lambda^2}\, t) + C_2 \sin(\sqrt{\omega^2 - \lambda^2}\, t)]$$



Fundamental frequency:

$$\omega_d = \sqrt{\omega^2 - \lambda^2} = \sqrt{\omega_n^2 - \lambda^2}$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$\lambda^2 - \omega^2 < 0$ → Under-damped system

Fundamental frequency: $\omega_d = \sqrt{\omega_n^2 - \lambda^2}$ -- see previous equation for $x(t)$

$$\omega_d = \sqrt{\omega_n^2 - \lambda^2} = \omega_n \sqrt{1 - \frac{\lambda^2}{\omega_n^2}} = \omega_n \sqrt{1 - \frac{\left(\frac{b}{2m}\right)^2}{\frac{k}{m}}}$$

Recall: critical damping coefficient:

$$b_c = 2m\sqrt{\frac{k}{m}} = 2m\,\omega_n = 2\sqrt{k\,m}$$

Un-damped fundamental frequency

$$\omega_d = \omega_n \sqrt{1 - \left(\frac{b}{b_c}\right)^2} = \omega_n \sqrt{1 - \zeta^2}$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$\lambda^2 - \omega^2 < 0$ → Under-damped system

Note that it is possible to write: $\lambda = \zeta\,\omega_n$ (Demonstrate in-class)

Solution of the governing differential equation can be written as:

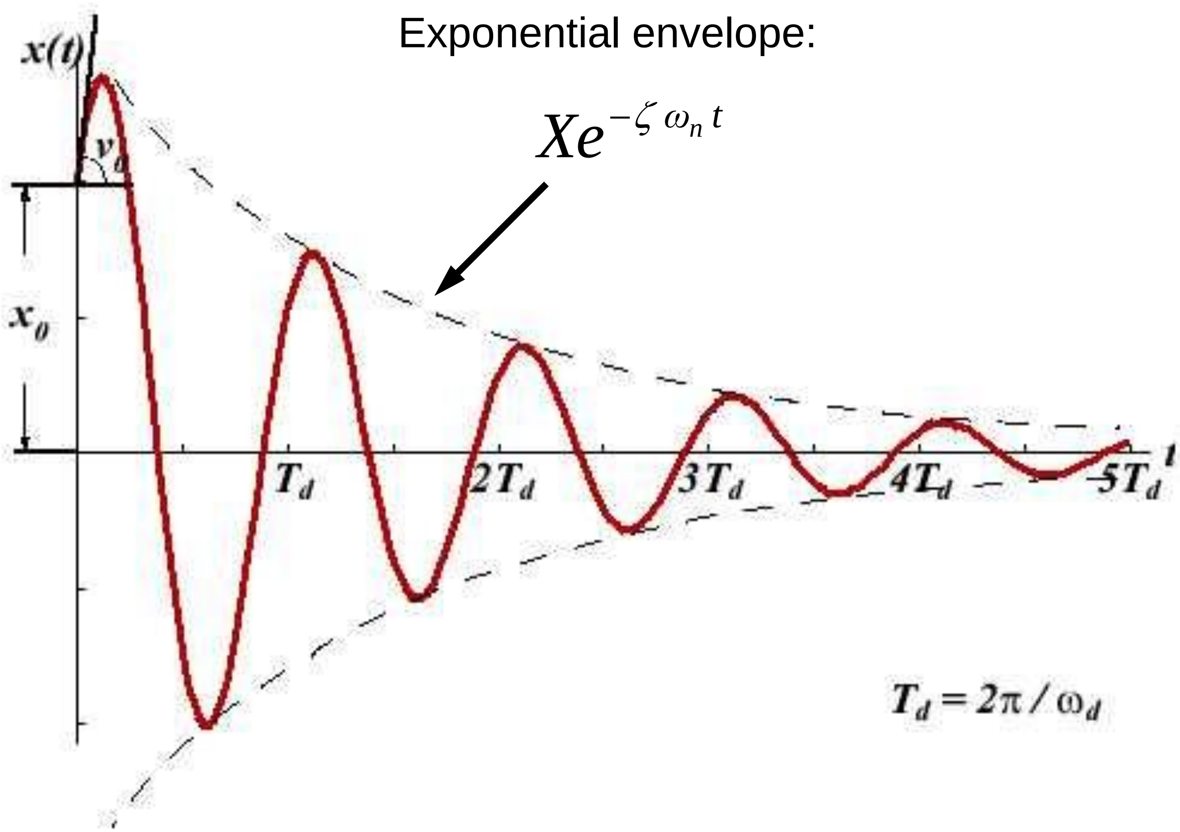
$$x(t) = e^{-\zeta\,\omega_n\,t}[C_1 \cos(\sqrt{1-\zeta^2}\,\omega_n\,t) + C_2 \sin(\sqrt{1-\zeta^2}\,\omega_n\,t)]$$

$$= X e^{-\zeta\,\omega_n\,t} \sin(\sqrt{1-\zeta^2}\,\omega_n\,t + \phi)$$

Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$\lambda^2 - \omega^2 < 0$ → Under-damped system

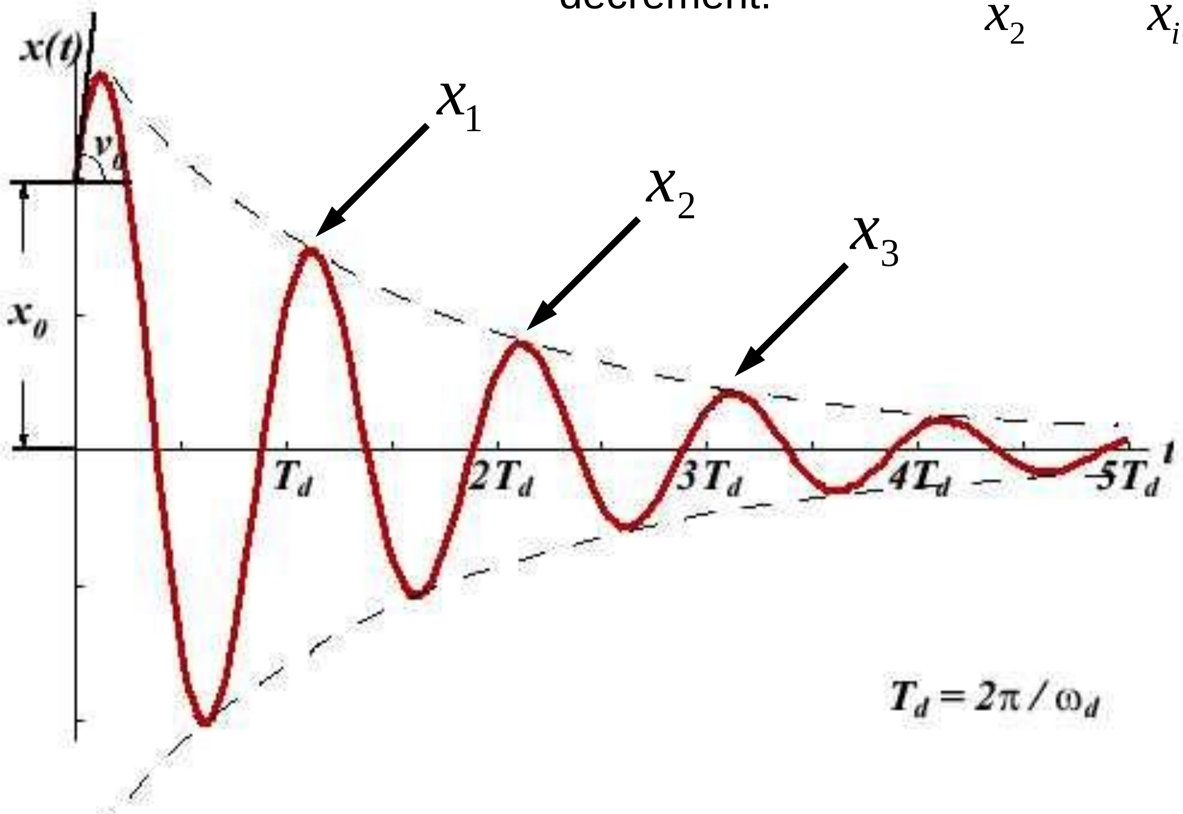


Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$\lambda^2 - \omega^2 < 0$ → Under-damped system

Logarithmic decrement: $$\delta = \ln \frac{x_1}{x_2} = \ln \frac{x_i}{x_{i+1}}$$



Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

$\lambda^2 - \omega^2 < 0$ → Under-damped system

Logarithmic decrement:

$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{e^{-\zeta\,\omega_n\,t_1}\sin(\sqrt{1-\zeta^2}\,\omega_n\,t_1 + \phi)}{e^{-\zeta\,\omega_n\,(t_1+T_d)}\sin[\sqrt{1-\zeta^2}\,\omega_n\,(t_1+T_d)+\phi]}$$

$$= \ln\frac{x_1}{x_2} = \ln\frac{e^{-\zeta\,\omega_n\,t_1}}{e^{-\zeta\,\omega_n\,(t_1+T_d)}} = \ln e^{\zeta\,\omega_n\,T_d} = \zeta\,\omega_n\,T_d$$

Recall that: $T_d = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \quad \Rightarrow \quad \boxed{\delta = \dfrac{2\pi\,\zeta}{\sqrt{1-\zeta^2}}}$