Worcester Polytechnic Institute
Mechanical Engineering Department

Engineering Experimentation
ME-3901, D’2012

Lecture 11
18 April 2012
General information

Office hours

Instructors: Cosme Furlong
Office: HL-151
Everyday:
9:00 to 9:50 am

Christopher Scarpino
Office: HL-153
During laboratory sessions

Teaching Assistants: During laboratory sessions
Equivalent systems

Original system

Equivalent system

\[ F_d = b \cdot \frac{dx}{dt} \]

\[ F_s = Kx \]

Thickness = 0.175"

(a) \( y = 4.0 \)"
(b) \( y = 5.5 \)"
(c) \( y = 7.0 \)"
(d) \( y = 8.5 \)"
(e) \( y = 10.0 \)"
(f) \( y = 11.5 \)"
(g) \( y = 18.125 \)"
Equivalent systems

Table of Spring Stiffness

\[ k = \frac{1}{1/k_1 + 1/k_2} \]

\[ k = k_1 + k_2 \]

\[ k = \frac{EI}{l} \quad I = \text{moment of inertia of cross-sectional area} \]

\[ k = \frac{EA}{l} \quad A = \text{cross-sectional area} \]

\[ k = \frac{GJ}{l} \quad J = \text{torsion constant of cross section} \]

\[ k = \frac{Gd^4}{64nR^3} \quad n = \text{number of turns} \]

Analysis of a single degree of freedom system

Governing equation:

\[ m \frac{d^2 x}{dt^2} = \sum_i F_i \]

\[ m \frac{d^2 x}{dt^2} = -kx - b \frac{dx}{dt} + F(t) \]
Analysis of a single degree of freedom system

First case: \( F(t) = 0 \) - Free vibrations

Consider governing equation:

\[
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0
\]

Second order differential equation with constant coefficients.

Governing equation can be written as:

\[
\frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0
\]

Possible solution has the form:

\[
x(t) = e^{m_i t}
\]

So the characteristic equation is:

\[
m_i^2 + 2\lambda m_i + \omega^2 = 0
\]

Roots of characteristic equation:

\[
m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}
\]
\[
m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}
\]
Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

\[ \lambda^2 - \omega^2 = 0 \quad \text{Critically damped system} \]

\[ \lambda^2 - \omega^2 > 0 \quad \text{Over-damped system} \]

\[ \lambda^2 - \omega^2 < 0 \quad \text{Under-damped system} \]
Analysis of a single degree of freedom system

First case: \( F(t) = 0 \) - Free vibrations

\[
\lambda^2 - \omega^2 = 0 \quad \text{Critically damped system}
\]

Solution to the governing differential equation:

\[
x(t) = C_1 e^{m_1 t} + C_2 t e^{m_1 t}
\]

Critical damping coefficient:

\[
b_c = 2m \sqrt{\frac{k}{m}} = 2m \omega_n = 2\sqrt{k/m}
\]

Fundamental frequency:

\[
\omega = \omega_n = \sqrt{\frac{k}{m}}
\]

Critical damping factor \( b_c \) is the minimum damping that results in non-periodic motions
Analysis of a single degree of freedom system

First case: \( F(t) = 0 \) - Free vibrations

\[ \lambda^2 - \omega^2 > 0 \quad \text{Over-damped system} \]

Solution to the governing differential equation:

\[ x(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t} \]

\[ x(t) = e^{-\lambda t \left( C_1 e^{\sqrt{\lambda^2 - \omega^2} t} + C_2 e^{-\sqrt{\lambda^2 - \omega^2} t} \right)} \]

Fundamental frequency:

\[ \omega = \omega_n = \sqrt{\frac{k}{m}} \]
Analysis of a single degree of freedom system

First case: \( F(t) = 0 \) - Free vibrations

\[ \lambda^2 - \omega^2 < 0 \quad \text{Under-damped system} \]

Solution to the governing differential equation:

\[ x(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t} \quad (m_1 \text{ and } m_2 \text{ are complex numbers, why?}) \]

\[ x(t) = e^{-\lambda t} \left[ C_1 \cos(\sqrt{\omega^2 - \lambda^2} \ t) + C_2 \sin(\sqrt{\omega^2 - \lambda^2} \ t) \right] \]

Fundamental frequency:

\[ \omega_d = \sqrt{\omega^2 - \lambda^2} = \sqrt{\omega_n^2 - \lambda^2} \]

\[ T_d = \frac{2\pi}{\omega_d} \]
Analysis of a single degree of freedom system

First case: \( F(t) = 0 \) - Free vibrations

\[ \lambda^2 - \omega^2 < 0 \quad \text{Under-damped system} \]

Fundamental frequency: \( \omega_d = \sqrt{\omega_n^2 - \lambda^2} \) -- see previous equation for \( x(t) \)

\[
\omega_d = \sqrt{\omega_n^2 - \lambda^2} = \omega_n \sqrt{1 - \frac{\lambda^2}{\omega_n^2}} = \omega_n \sqrt{1 - \frac{\left(\frac{b}{2m}\right)^2}{k/m}}
\]

Recall: critical damping coefficient:

\[ b_c = 2m \sqrt{\frac{k}{m}} = 2m \omega_n = 2\sqrt{km} \]

Un-damped fundamental frequency

\[ \omega_d = \omega_n \sqrt{1 - \left(\frac{b}{b_c}\right)^2} = \omega_n \sqrt{1 - \zeta^2} \]
Analysis of a single degree of freedom system

First case: \( F(t) = 0 \) - Free vibrations

\[ \lambda^2 - \omega^2 < 0 \quad \rightarrow \quad \text{Under-damped system} \]

Note that it is possible to write: \( \lambda = \zeta \omega_n \) (Demonstrate in-class)

Solution of the governing differential equation can be written as:

\[
x(t) = e^{-\zeta \omega_n t} \left[ C_1 \cos(\sqrt{1 - \zeta^2} \, \omega_n \, t) + C_2 \sin(\sqrt{1 - \zeta^2} \, \omega_n \, t) \right]
\]

\[ = Xe^{-\zeta \omega_n t} \sin(\sqrt{1 - \zeta^2} \, \omega_n \, t + \phi) \]
Analysis of a single degree of freedom system

First case: \( F(t) = 0 \) - Free vibrations

\[
\lambda^2 - \omega^2 < 0 \quad \rightarrow \quad \text{Under-damped system}
\]

Exponential envelope:

\[
Xe^{-\zeta \omega_n t}
\]

\[T_d = \frac{2\pi}{\omega_d}\]
Analysis of a single degree of freedom system

First case: $F(t) = 0$ - Free vibrations

\[ \lambda^2 - \omega^2 < 0 \quad \text{Under-damped system} \]

Logarithmic decrement:

\[ \delta = \ln \frac{x_1}{x_2} = \ln \frac{x_i}{x_{i+1}} \]

\[ T_d = \frac{2\pi}{\omega_d} \]
Analysis of a single degree of freedom system

First case: \( F(t) = 0 \) - Free vibrations

\[ \lambda^2 - \omega^2 < 0 \quad \rightarrow \quad \text{Under-damped system} \]

Logarithmic decrement:

\[
\delta = \ln \frac{x_1}{x_2} = \ln \frac{e^{-\zeta \omega_n t_1} \sin(\sqrt{1 - \zeta^2} \omega_n t_1 + \phi)}{e^{-\zeta \omega_n (t_1 + T_d)} \sin[\sqrt{1 - \zeta^2} \omega_n (t_1 + T_d) + \phi]}
\]

\[
= \ln \frac{x_1}{x_2} = \ln \frac{e^{-\zeta \omega_n t_1}}{e^{-\zeta \omega_n (t_1 + T_d)}} = \ln e^{\zeta \omega_n T_d} = \zeta \omega_n T_d
\]

Recall that: \( T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}} \quad \Rightarrow \quad \delta = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \)