Worcester Polytechnic Institute
Mechanical Engineering Department

Engineering Experimentation
ME-3901, A’2010

Lecture 09
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General information

Office hours

**Instructor:** Cosme Furlong; cfurlong@wpi.edu
Everyday from 11:00 to 11:50 am
or by appointment

**Teaching Assistant:** Jeffrey Laut & Kazim Naqvi;
During Lab Sessions
Strain gages

Definition of gage factor:

\[ F = \frac{dR}{R} \frac{1}{\varepsilon_a} \]

(From previous discussion)

\[ \Rightarrow F = 1 + 2\mu + \frac{1}{\varepsilon_a} \frac{d\rho}{\rho} \]

If resistivity does not change

\[ \Rightarrow F = 1 + 2\mu \]

And strain with change of resistance is:

\[ \Rightarrow \varepsilon_a = \frac{1}{F} \frac{\Delta R}{R} \]

A typical strain gage has a gage factor \( \approx 2.095 \pm 0.5\% \). Why? How is this possible? Open for discussions
Strain gages and a Wheatstone bridge

Recall from previous discussions:
(Changes in resistance & output voltage)

\[ \frac{\Delta E_g}{E} \approx \frac{\Delta R_4}{4R} = \frac{\Delta R}{4R} \]

And strain with change of resistance is:

\[ \varepsilon_a = \frac{1}{F} \frac{\Delta R}{R} \]

We want to recover strain from voltage measurements.
Combine previous equations:

\[ \varepsilon_a = \frac{1}{F} \frac{4\Delta E_g}{E} \]
Strain gages and a Wheatstone bridge
We need to amplify output signal: determine gain

Re-write previous equation as:

\[ \Delta E_g = \frac{F}{4} \cdot E \cdot \varepsilon_a \]

Assume the following values:
(based on an actual setup)

\[ E = 10 \pm 0.005 \text{ V} \]
\[ F = 2.095 \pm 0.5\% \]

Also, assume the measurement of
only 1 \(\mu\)strain (\(\varepsilon\mu\)):

\[ \varepsilon_a = 1 \mu\text{strain} = 1 \times 10^{-6} \]

Using these values leads to:

\[ \Delta E_g = 5.238 \times 10^{-6} \text{ V} \]

Is it possible to measure this voltage level in HL-031?
Open for discussions
Strain gages and a Wheatstone bridge

We need to amplify output signal: determine gain

Assume that measurement resolution of DAQ system is:
(please, update accordingly, while taking into account max./min. voltages allowed in the DAQs input)

\[
\text{Gain for the output signal should be: } \quad \text{Gain} = \frac{1 \times 10^{-3} V}{5.238 \times 10^{-6} V} \approx 191
\]

If we use max. meas. resolution of DAQs in HL-031, what is the range of strain values that can be measured?

Open for discussions
Note on strain gages design (typical: 0.001” thick)

Resistance = $R = \rho \frac{L}{A}$

Gage factor = $F = \frac{dR}{R} / \varepsilon_x$

$\varepsilon_x = \frac{1}{F} \frac{\Delta R}{R}$

To be measured (use a bridge circuit)
Strain gages and a Wheatstone bridge

Calibration by use of shunt resistors

Connect high-precision calibration resistor in “parallel”

\[ R_{\text{cal}} = R_{\text{shunt}} \]
Strain gages and a Wheatstone bridge
Calibration by use of shunt resistors

Measuring arm of the bridge

\[ E_4 \quad R_x \quad R_{cal} = R_{shunt} \]

Calibration resistor
Strain gage
Strain gages and a Wheatstone bridge
Calibration by use of shunt resistors

Equivalent resistance

\[
\frac{1}{R} = \frac{1}{R_x} + \frac{1}{R_{cal}} \quad \Rightarrow \quad R = \frac{R_x \cdot R_{cal}}{R_x + R_{cal}}
\]
Strain gages and a Wheatstone bridge
Calibration by use of shunt resistors

Change in resistance is

\[ \Delta R = R - R_x = \frac{R_x \cdot R_{cal}}{R_x + R_{cal}} - R_x \]

\[ = - \frac{R_x^2}{R_x + R_{cal}} \]
Strain gages and a Wheatstone bridge
Calibration by use of shunt resistors

Using the definition of a gage factor:

\[ \varepsilon_{cal} = \frac{\Delta R}{F R_x} \quad \Rightarrow \quad \varepsilon_{cal} = -\frac{R_x}{F (R_x + R_{cal})} \]

Indicates compression
Strain gages and a Wheatstone bridge
Calibration by use of shunt resistors

Example

If: \( R_{cal} = 878,000 \, \Omega \); \( R_x = 120 \, \Omega \) with \( F = 2.095 \)

\[ \varepsilon_{cal} = -\frac{R_x}{F (R_x + R_{cal})} = -\frac{120}{2.095(120 + 878,000)} = -65.2 \times 10^{-6} \]

\[ = -65.2 \, \mu \text{strain (compression)} \]
Strain gages and a Wheatstone bridge
Calibration by use of shunt resistors

Amplifier model 2310

- Internal calibration resistors
- Internal variable resistor (bridge calibration)
- Gain (note resolution in gain settings)
Strain gages and a Wheatstone bridge
Calibration by use of shunt resistors

Amplifier model 2310 in \( \frac{1}{4} \) bridge configuration

+ A: 59.94 k\( \Omega \) \( \Rightarrow \) \( \approx \) 1000 \( \mu \)strain
+ B: 174.8 k\( \Omega \) \( \Rightarrow \) \( \approx \) 340 \( \mu \)strain

Check + + - and \( \varepsilon_{\text{cal}} \)
Strain gages and a Wheatstone bridge
Calibration by use of shunt resistors

+ A resistor

VI built for the lab
Strain gages and a Wheatstone bridge
Calibration by use of shunt resistors

+ B resistor

VI built for the lab
Use of strains to compute pressure and stresses

Governing equation (based on tangential - Hoop - strain):

\[
P = \frac{E t \varepsilon}{r (1 - \nu / 2)} \quad \Rightarrow \quad P = P(E, t, \varepsilon, r, \nu)
\]

Pressure is obtained from these equations:

\[
\varepsilon_{Hoop} = \varepsilon = \frac{1}{E} (\sigma_{Hoop} - \nu \sigma_{Axial})
\]

\[
\sigma_{Hoop} = \frac{P r}{t} ; \quad \sigma_{Axial} = \frac{P r}{2t}
\]
Determination of principal stresses

Principal **normal** stresses

- This problem involves performing coordinate transformation, which can provide a stress tensor that does NOT contain shear stresses.
- In 2D, this can be illustrated as:

  Stress cube in original coordinate system (x,y)

  Stress cube in transformed coordinate system (x',y') -- only normal stresses exist: $\sigma_1$ and $\sigma_3$, in this 2D case.
Determination of principal stresses
Principal \textit{normal} stresses

This problem involves performing coordinate transformation, which can provide a stress tensor that does NOT contain shear stresses, that is:

\[
\begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{yz} & \sigma_{zz}
\end{bmatrix}
\begin{bmatrix}
\sigma \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
\sigma \\
0 \\
0 \\
0 \\
\sigma \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\sigma \\
0 \\
0 \\
0 \\
\sigma \\
0 \\
0
\end{bmatrix}
\]

\(\hat{n}\)}

Unit vector, normal to principal plane

Initial stress tensor

Transformed stress tensor

Same vectors
Determination of principal stresses

Principal *normal* stresses

Previous equation can be written as

\[
\begin{bmatrix}
\sigma_{xx} - \sigma & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} - \sigma & \tau_{yz} \\
\tau_{zx} & \tau_{yz} & \sigma_{zz} - \sigma
\end{bmatrix}
\hat{\mathbf{n}} = 
\begin{bmatrix}
\sigma_{xx} - \sigma & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} - \sigma & \tau_{yz} \\
\tau_{zx} & \tau_{yz} & \sigma_{zz} - \sigma
\end{bmatrix}
\begin{pmatrix}
\mathbf{n}_x \\
\mathbf{n}_y \\
\mathbf{n}_z
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

implying that the determinant

\[
\begin{vmatrix}
\sigma_{xx} - \sigma & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} - \sigma & \tau_{yz} \\
\tau_{zx} & \tau_{yz} & \sigma_{zz} - \sigma
\end{vmatrix} = 0
\]
Determination of principal stresses

Principal normal stresses

Expanding determinant and setting it to zero yields

\[ \sigma^3 - C_2 \sigma^2 + C_1 \sigma - C_0 = 0 \]

in which

\[ C_2 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \]
\[ C_1 = \sigma_{xx} \sigma_{yy} + \sigma_{yy} \sigma_{zz} + \sigma_{zz} \sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \]
\[ C_0 = \sigma_{xx} \sigma_{yy} \sigma_{zz} + 2\tau_{xy} \tau_{yz} \tau_{zx} - \sigma_{xx} \tau_{yz}^2 - \sigma_{yy} \tau_{zx}^2 - \sigma_{zz} \tau_{xy}^2 \]

are stress invariants (have the same magnitudes for all choices of coordinate axes \((x,y,z)\) in which the applied stresses are measured or calculated.)

The principal normal stresses, \( \sigma_1, \sigma_2, \sigma_3 \), are the three roots of the cubic polynomial -- always real and typically ordered as: \( \sigma_1 > \sigma_2 > \sigma_3 \)
Determination of principal stresses

Principal shear stresses

Principal shear stresses can be found from values of the principal normal stresses as

\[ \tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2} \]

\[ \tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2} \]

\[ \tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2} \]
Determination of principal stresses

Principal normal and shear stresses: 2D case

These equations are used extensively

Principal normal stresses:

\[
\sigma_1, \sigma_3 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}
\]

Maximum shear stress:

\[
\tau_{\text{Max}} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}
\]
Mohr’s circle: principal normal and shear stresses

Graphical representation of previous equations: 3D

Fig. 1-4.5/Mohr’s Circle in three dimensions.
From Boresi: Mechanics of materials
Reading assignment

- Beckwith: Ch. 7, 12, Appendix E
- Bishop: Ch. 11

References:

Homework assignment: Handout-H

- Beckwith: 12.7, 12.10, 12.11
- Bishop: Section 11.3