WORCESTER POLYTECHNIC INSTITUTE

MECHANICAL ENGINEERING DEPARTMENT

Engineering Experimentation
ME-3901, D'2012

Lecture 04

21 March 2012

General information

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During laboratory sessions

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Curve fitting: introduction

The most common and most efficient ways to present quantitative data are by the use of tables, plots (figures), and mathematical equations. Which method of presentation is the most appropriate form of reporting depends on a variety of factors, which include the purpose of the presentation, the quantity of data, the degree of precision of the results, and the number of variables. Because so many factors are involved, only a few general remarks can be made. Your choice of the particular form of presentation must be made largely by applying common sense guided by experience.

Table 6.1.

X	Y
10.0	8.04
8.0	6.95
13.0	7.58
9.0	8.81
11.0	8.33
14.0	9.96
6.0	7.24
4.0	4.26
12.0	10.84
7.0	4.82
5.0	5.68

Table 6.2.

X	Y
10.0	9.14
8.0	8.14
13.0	8.74
9.0	8.77
11.0	9.26
14.0	8.10
6.0	6.13
4.0	3.10
12.0	9.13
7.0	7.26
5.0	4.74

Table 6.3.

X	Y
10.0	7.46
8.0	6.77
13.0	12.74
9.0	7.11
11.0	7.81
14.0	8.84
6.0	6.08
4.0	5.39
12.0	8.15
7.0	6.42
5.0	5.73

Table 6.4.

X	Y
8.0	6.58
8.0	5.76
8.0	7.71
8.0	8.84
8.0	8.47
8.0	7.04
8.0	5.25
19.0	12.50
8.0	5.56
8.0	7.91
8.0	6.89

For example, four sets of data are shown in Tables 6.1 to 6.4, respectively. Each one of these data sets has exactly the same mean value of X's = 9.0 and of Y's = 7.5. Their standard deviations are also exactly the same and equal to $\sigma_x = 3.16$ and $\sigma_y = 1.94$. Furthermore, each of these four data sets is represented by the same equation of a straight line, obtained by the least-squares analysis, which is

$$Y = 0.50X + 3.00 \quad , \tag{6.1}$$

and which results in a correlation coefficient of 0.77, for each data set. However, graphical representations of the four data sets are drastically different, as shown in Figs

6.1 to 6.4, respectively. This example represents the so called **Anscombe's quartet**[1] and is a classical justification of the need for careful data presentation.

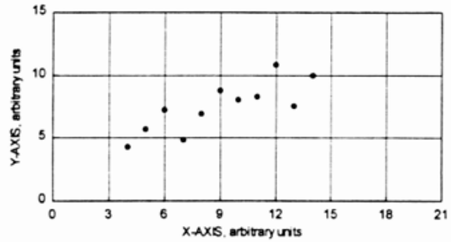

Fig. 6.1. Graphical representation of the data presented in Table 6.1.

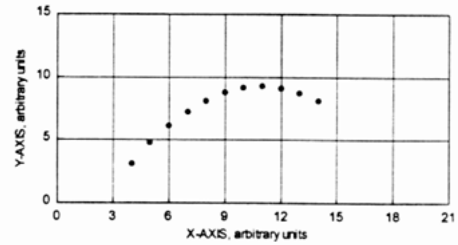

Fig. 6.2. Graphical representation of the data presented in Table 6.2.

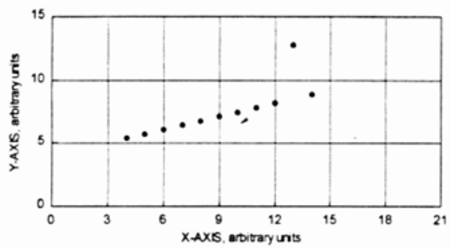

Fig. 6.3. Graphical representation of the data presented in Table 6.3.

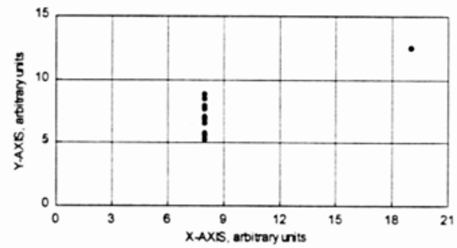

Fig. 6.4. Graphical representation of the data presented in Table 6.4.

The foregoing discussion indicates that mere tabulation of the data and computation of statistical parameters representing these data are not adequate for meaningful presentation of results. **Graphical representation of the results is extremely important and aided with statistics and mathematical equations provides complete presentation of data.** The subject of this note is graphical, statistical, and mathematical presentation of data and is illustrated with a representative problem.

6.2. Method of least squares

Table 6.1 represents variables X and Y over a range of values. What is needed is a mathematical expression that will represent Y as a function of X. The simplest type of function is a linear one, which establishes Y as a linear function of X expressed as

$$Y = aX + b \quad , \tag{6.2}$$

where a is the slope of the straight line and b is the intercept, as illustrated in Fig. 6.5. The task to be performed is to find the linear function that will best represent the given data. This is achieved by first defining the error, E_i, for an individual data point i, as

$$E_i = Y_i - (aX_i + b) \quad . \tag{6.3}$$

and then by defining the sum, S, of all the errors squared as

$$S = \sum_{i=1}^{n} (E_i)^2 = \sum_{i=1}^{n} [Y_i - (aX_i + b)]^2 \quad . \tag{6.4}$$

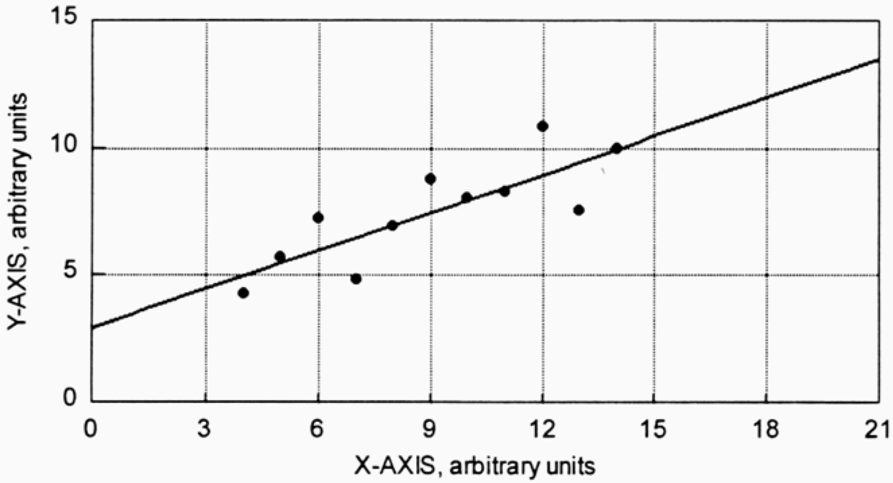

Fig. 6.5. An example of a linear least-squares curve fit to the data shown in Fig. 6.1.

Equation 6.4 is minimized by setting the derivatives of S with respect to a and b equal to zero, that is,

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} 2[Y_i - (aX_i + b)](-X_i) = 0 \quad , \tag{6.5}$$

and

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} 2[Y_i - (aX_i + b)](-1) = 0 \quad . \tag{6.6}$$

Simplification of Eqs. 6.5 and 6.6 results in a system of two simultaneous equations

$$2\sum_{i=1}^{n}[(aX_i + b)X_i] = 2\sum_{i=1}^{n}(Y_iX_i \quad , \tag{6.7}$$

and

$$2\sum_{i=1}^{n}(aX_i + b) = 2\sum_{i=1}^{n}Y_i \quad , \tag{6.8}$$

which can be rewritten as

$$a\sum_{i=1}^{n}(X_i)^2 + b\sum_{i=1}^{n}(X_i) = \sum_{i=1}^{n}(Y_iX_i \quad , \tag{6.9}$$

and

$$a\sum_{i=1}^{n}(X_i) + bn = \sum_{i=1}^{n}(Y_i \quad . \tag{6.10}$$

Equations 6.9 and 6.10 can be written in a matrix form as

$$\begin{bmatrix} \sum_{i=1}^{n} X_i^2 & \sum_{i=1}^{n} X_i \\ \sum_{i=1}^{n} X_i & n \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} X_iY_i \\ \sum_{i=1}^{n} Y_i \end{pmatrix} \quad . \tag{6.11}$$

Solution of Eq. 6.11 yields

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} \sum_{i=1}^{n} X_i^2 & \sum_{i=1}^{n} X_i \\ \sum_{i=1}^{n} X_i & n \end{bmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{n} X_iY_i \\ \sum_{i=1}^{n} Y_i \end{pmatrix} \quad . \tag{6.12}$$

or

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sum_{i=1}^{n}(X_i)^2 - \left(\sum_{i=1}^{n} X_i\right)^2} \begin{bmatrix} n & -\sum_{i=1}^{n} X_i \\ -\sum_{i=1}^{n} X_i & \sum_{i=1}^{n}(X_i)^2 \end{bmatrix} \begin{pmatrix} \sum_{i=1}^{n} X_iY_i \\ \sum_{i=1}^{n} Y_i \end{pmatrix} \quad . \tag{6.13}$$

Equation 6.13, may be written in a more conventional forms as

$$a = \frac{n\sum_{i=n}^{n}(X_iY_i) - \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)}{n\sum_{i=1}^{n}\left(X_i^2\right) - \left(\sum_{i=1}^{n} X_i\right)^2} \quad , \tag{6.14}$$

and

$$b = \frac{\left(\sum_{i=1}^{n} Y_i\right)\left(\sum_{i=1}^{n} X_i^2\right) - \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} X_i Y_i\right)}{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2} \quad . \qquad (6.15)$$

Equations 6.14 and 6.15 give values of coefficients a and b which best fit the given data set. Following the procedure used above, equations for constants corresponding to higher order curve fits can be determined.

6.3. General form of least squares

The method shown in Section 6.2 is specific to linear, or first order, least squares curve fits. It should be noted that any order polynomial can be fit to data, provided there are more data points than the order of the polynomial (i.e. four points are needed to perform a third order curve fit).

To fit n data points to a general m^{th} order polynomial given by

$$Y = A_1 X^m + A_2 X^{m-1} + \ldots + A_m X + A_{m+1} \quad , \qquad (6.16)$$

an expanded form of the matrix given by Eq. 6.11 is used. This matrix takes the general form of

$$\begin{bmatrix} \sum_{i=1}^{n} X_i^{2m} & \sum_{i=1}^{n} X_i^{2m-1} & \sum_{i=1}^{n} X_i^{2m-2} & \cdots & \sum_{i=1}^{n} X_i^{m} \\ \sum_{i=1}^{n} X_i^{2m-1} & \sum_{i=1}^{n} X_i^{2m-2} & \ddots & \ddots & \vdots \\ \sum_{i=1}^{n} X_i^{2m-2} & \ddots & \ddots & \ddots & \sum_{i=1}^{n} X_i^{2} \\ \vdots & \ddots & \ddots & \sum_{i=1}^{n} X_i^{2} & \sum_{i=1}^{n} X_i \\ \sum_{i=1}^{n} X_i^{m} & \cdots & \sum_{i=1}^{n} X_i^{2} & \sum_{i=1}^{n} X_i & n \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \\ A_{m+1} \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} X_i^{m} Y_i \\ \sum_{i=1}^{n} X_i^{m-1} Y_i \\ \vdots \\ \sum_{i=1}^{n} X_i Y_i \\ \sum_{i=1}^{n} Y_i \end{Bmatrix} \quad , \qquad (6.17)$$

when considering the case of the polynomial given by Eq. 6.16.

To illustrate this general least squares approach, consider the 3rd order polynomial given by

$$Y = aX^3 + bX^2 + cX + d \quad . \qquad (6.18)$$

To solve for the four coefficients a, b, c, and d, Eq. 6.17 is applied to Eq. 6.18 and results in

$$\begin{bmatrix} \sum_{i=1}^{n} X_i^6 & \sum_{i=1}^{n} X_i^5 & \sum_{i=1}^{n} X_i^4 & \sum_{i=1}^{n} X_i^3 \\ \sum_{i=1}^{n} X_i^5 & \sum_{i=1}^{n} X_i^4 & \sum_{i=1}^{n} X_i^3 & \sum_{i=1}^{n} X_i^2 \\ \sum_{i=1}^{n} X_i^4 & \sum_{i=1}^{n} X_i^3 & \sum_{i=1}^{n} X_i^2 & \sum_{i=1}^{n} X_i \\ \sum_{i=1}^{n} X_i^3 & \sum_{i=1}^{n} X_i^2 & \sum_{i=1}^{n} X_i & n \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} X_i^3 Y_i \\ \sum_{i=1}^{n} X_i^2 Y_i \\ \sum_{i=1}^{n} X_i Y_i \\ \sum_{i=1}^{n} X_i \end{Bmatrix} . \qquad (6.19)$$

6.4. Example problems

6.4.1. Problem statement

A set of data, representing object's velocity measured as a function of time, is given in Table 6.5.

1) represent these graphically on a T-t (i.e., velocity versus time) diagram.

2) determine mathematical formula , where V is velocity, t is time, and a and b are constants to be determined, for the data of Table 6.5, using the method of least squares. Show all work.

3) compute errors, E, by determining differences between measured and curve fitted points.

4) plot results of step 3 on an E-t diagram.

5) discuss your results.

Table 6.5. Data set representing object's velocity measured as a function of time.

t (sec)	V (ft/sec)
0	1.45
1	2.60
2	3.60
3	4.50
4	4.95
5	5.20
6	5.10
7	4.75
8	4.10

6.4.1.1. Solution

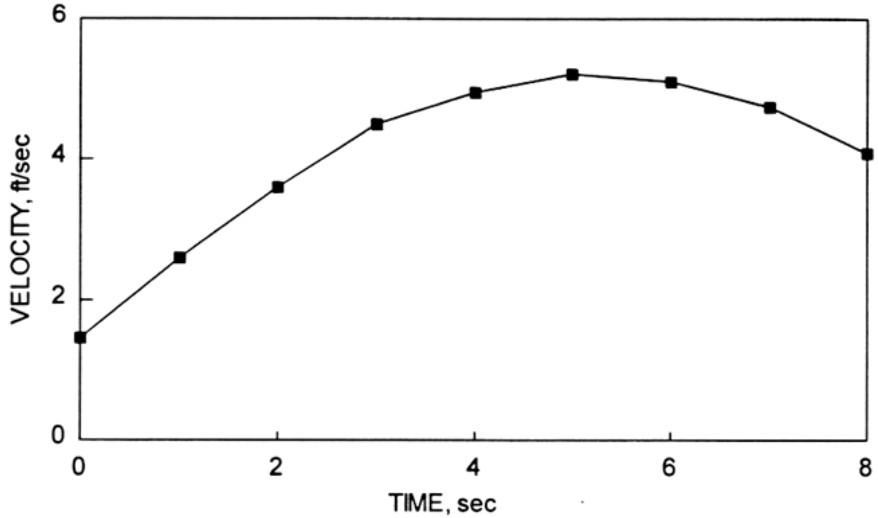

Fig. 6.6. Graphical representation of data in Table 6.5.

To determine the first order line which best fits the nine points in Table 6.1, several summations need to be computed. These summations are

$$\sum_{i=1}^{9} X_i = 0+1+2+3+4+5+6+7+8 = 36 \quad ,$$

$$\sum_{i=1}^{9} X_i^2 = 0+1+4+9+16+25+36+49+64 = 204 \quad ,$$

, and

$$\sum_{i=1}^{9} X_i Y_i = 0+2.60+7.20+13.50+19.80+26.00+30.60+33.25+32.80$$

$$= 165.75 \quad . \tag{6.20}$$

These values are inserted into Eqs. 6.9 and 6.10 to produce

$$a(204)+b(36) = (165.75) \quad , \tag{6.21}$$

and

$$a(36) + b(9) = (36.25) \quad . \tag{6.22}$$

Solving Eqs. 6.21 and 6.22 for a and b results in values of $a = 0.3458$ and $b = 2.6442$. Using these values in Eq. 6.2 produces the equation for a linear least squares curve fit of the points in Table 6.5. This resultant equation is

$$Y = 0.3458\ X + 2.6442 \quad . \tag{6.23}$$

Equation 6.23 can be used to compute the an fitted set of points corresponding to those listed in Table 6.5. These fitted points are given in Table 6.6.

Table 6.6. Data set representing object's velocity measured as a function of time shown with data from first order least squares curve fit.

t (sec)	V [data] (ft/sec)	V [fitted] (ft/sec)	Difference (ft/sec)
0	1.45	2.64	-1.19
1	2.60	2.99	-0.39
2	3.60	3.34	0.26
3	4.50	3.68	0.82
4	4.95	4.03	0.92
5	5.20	4.37	0.83
6	5.10	4.72	0.38
7	4.75	5.06	-0.31
8	4.10	5.41	-1.31

6.4.2. Problem statement

A set of data representing an object's velocity measured as a function of time, is given in Table 6.5.

1) Determine mathematical formula $V = V(t) = a\,t^2 + b\,t + c$, where V is velocity, t is time, and a, b, and c are constants to be determined, for the data Table 6.1, using the method of least squares. Show all work.

2) Compute errors, E, by determining differences between measured and curve fitted points.

3) Plot results of step 3 on an E-t diagram.

4) Compare errors found in this Problem with the errors of Section 6.4 Part 4. Discuss your observations.

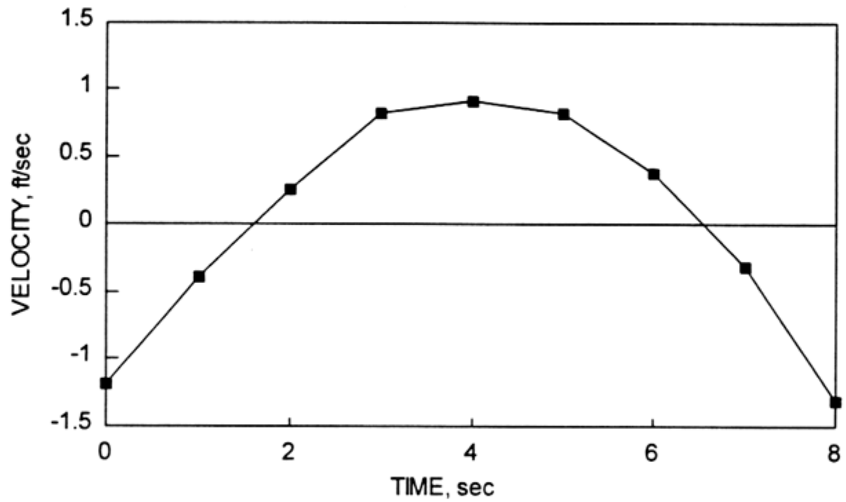

Fig. 6.7. Graphical representation of the data in Table 6.6 showing the difference between the measured points and the curve fitted points.

To perform the second order curve fit, three additional summations are needed in addition to those presented as Eq. 6.20. These summations are

$$\sum_{i=1}^{9} X_i^3 = 0+1+8+27+64+125+216+343+512 = 1296 \quad ,$$

$$\sum_{i=1}^{9} X_i^4 = 0+1+16+81+256+625+1296+2401+4096 = 8772 \quad , \quad \text{and}$$

$$\sum_{i=1}^{9} X_i^2 Y_i = 0+2.60+14.4+40.5+79.2+130.0+183.6+232.75+262.4$$

$$= 945.45 \quad . \tag{6.24}$$

The polynomial to which the data is to be fit is in the form of

$$Y = a\,X^2 + b\,X + c \quad . \tag{6.25}$$

The three simultaneous equations to be used to fit the measured data to Eq. 6.25 are

$$a\sum_{i=1}^{9} X_i^4 + b\sum_{i=1}^{9} X_i^3 + c\sum_{i=1}^{9} X_i^2 = \sum_{i=1}^{9} X_i^2 Y \quad , \tag{6.27}$$

$$a \sum_{i=1}^{9} X_i^3 + b \sum_{i=1}^{9} X_i^2 + c \sum_{i=1}^{9} X_i = \sum_{i=1}^{9} X_i Y \quad \text{, and} \tag{6.28}$$

$$a \sum_{i=1}^{9} X_i^2 + b \sum_{i=1}^{9} X_i + c \sum_{i=1}^{9} 1 = \sum_{i=1}^{9} \quad . \tag{6.29}$$

By putting the values from Eqs. 6.20 and 6.24 into the Eqs. 6.27 to 6.29, the relations

$$a(8772) + b(1296) + c(204) = 945.45 \quad , \tag{6.30}$$

$$a(1296) + b(204) + c(36) = 165.75 \quad \text{, and} \tag{6.31}$$

$$a(204) + b(36) + c(9) = 36.25 \quad . \tag{6.32}$$

Solving Eqs. 6.30 to 6.32 for a, b, and c, the values $a = -0.13706$, $b = 1.442370$, and $c = 1.365151$ are obtained. Substituting these values into Eq. 6.25, the equation representing the second order curve fit of the points given in Table 6.5 is obtained. This equation is

$$Y = -0.13706\, X^2 + 1.44237\, X + 1.36515 \quad . \tag{6.33}$$

Equation 6.33 can be used to compute the an fitted set of points corresponding to those listed in Table 6.5. These fitted points are given in Table 6.7.

Table 6.7. Data set representing object's velocity measured as a function of time shown with data from second order least squares curve fit.

t (sec)	V [data] (ft/sec)	V [fitted] (ft/sec)	Difference (ft/sec)
0	1.45	1.37	0.08
1	2.60	2.67	-0.07
2	3.60	3.70	-0.10
3	4.50	4.46	0.04
4	4.95	4.94	0.01
5	5.20	5.15	0.05
6	5.10	5.09	0.01
7	4.75	4.75	0.00
8	4.10	4.13	-0.03

Note that the differences shown in Table 6.7 and substantially smaller than those presented in Table 6.6. Also notice that the data has a somewhat random appearance in Fig. 6.8 while in Fig. 6.7 the data follows a distinct pattern. All this evidence points to the conclusion that a second order curve fit of the data is far better than a first order curve fit.

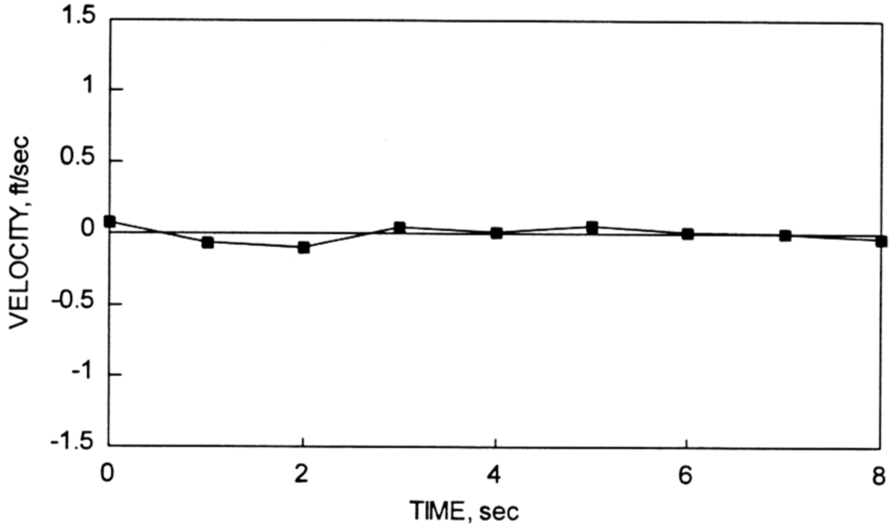

Fig. 6.8. Graphical representation of the data in Table 6.7 showing the difference between the measured points and the curve fitted points.

References

Anscombe, F. J., “Graphs in statistical analysis,” American Statistician, 27:17-21, 1973.

Pryputniewicz, R. J., “Engineering Experimentation,” Worcester Polytechnic Institute, 1993.

Curve fitting: matrix approach

Discussed in class: estimation of ± ranges for fitting coefficients

Curve fitting can be expressed as:

$$[X'X]\beta = (X'Y) \qquad ([X'] \text{ is transpose of } [X])$$

Obtained from model: $[X]\beta = (Y)$ (Over determined system of equations)

Vector of unknowns (coefficients of model used, e.g., slope and dc values in a linear fit)

Solution is: $\beta = [X'X]^{-1}(X'Y)$

Components of inverse matrix are expressed as:

$$[X'X]^{-1} = [c_{ij}]$$

Curve fitting: matrix approach

Discussed in class: estimation of ± ranges for fitting coefficients

Recall that sum of the square of the errors (SCE) is:

$$SCE = \sum_i (y_i - \hat{y}_i)^2$$ (also called "residuals" squared)

Measured point (y_i)

Corresponding curve-fitted point ($\hat{y}_i$)

Definition: estimator of variance:

$$S^2 = \left(\frac{1}{n - n_{\# parameters \beta}} \right) SCE$$

Number of degrees of freedom ($n - n_{\# parameters \beta}$)

$\# parameters \beta$ ($= 2$ for a line fit, i.e., β_0 and β_1 for $Y = \beta_0 + \beta_1 X$)

$n =$ number of points used in curve fitting

Curve fitting: matrix approach

Discussed in class: estimation of ± ranges for fitting coefficients

Confidence interval $(1-\alpha)100\%$ for fitting coefficient β_i :

$$\beta_i \pm t_{\alpha/2} S \sqrt{c_{ii}}$$

Student's $t-$distribution for the determined degrees of freedom (see previous page)

Interval for β_i fitting coefficients where their values are $(1-\alpha)100\%$ reliable

Curve fitting: matrix approach

Discussed in class: estimation of ± ranges for fitting coefficients

TABLE of CRITICAL VALUES for STUDENT'S *t* DISTRIBUTIONS

Column headings denote probabilities (α) ***above*** tabulated values.

degrees of freedom

d.f.	0.40	0.25	0.10	0.05	0.04	0.025	0.02	0.01	0.005	0.0025	0.001	0.0005
1	0.325	1.000	3.078	6.314	7.916	12.706	15.894	31.821	63.656	127.321	318.289	636.578
2	0.289	0.816	1.886	2.920	3.320	4.303	4.849	6.965	9.925	14.089	22.328	31.600
3	0.277	0.765	1.638	2.353	2.605	3.182	3.482	4.541	5.841	7.453	10.214	12.924
4	0.271	0.741	1.533	2.132	2.333	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.267	0.727	1.476	2.015	2.191	2.571	2.757	3.365	4.032	4.773	5.894	6.869
6	0.265	0.718	1.440	1.943	2.104	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.263	0.711	1.415	1.895	2.046	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.262	0.706	1.397	1.860	2.004	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.261	0.703	1.383	1.833	1.973	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.260	0.700	1.372	1.812	1.948	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.260	0.697	1.363	1.796	1.928	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.259	0.695	1.356	1.782	1.912	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.259	0.694	1.350	1.771	1.899	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.258	0.692	1.345	1.761	1.887	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.258	0.691	1.341	1.753	1.878	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.258	0.690	1.337	1.746	1.869	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.257	0.689	1.333	1.740	1.862	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.257	0.688	1.330	1.734	1.855	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.257	0.688	1.328	1.729	1.850	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.257	0.687	1.325	1.725	1.844	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.257	0.686	1.323	1.721	1.840	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.256	0.686	1.321	1.717	1.835	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.256	0.685	1.319	1.714	1.832	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.256	0.685	1.318	1.711	1.828	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.256	0.684	1.316	1.708	1.825	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.256	0.684	1.315	1.706	1.822	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.256	0.684	1.314	1.703	1.819	2.052	2.158	2.473	2.771	3.057	3.421	3.689
28	0.256	0.683	1.313	1.701	1.817	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.256	0.683	1.311	1.699	1.814	2.045	2.150	2.462	2.756	3.038	3.396	3.660
30	0.256	0.683	1.310	1.697	1.812	2.042	2.147	2.457	2.750	3.030	3.385	3.646
31	0.256	0.682	1.309	1.696	1.810	2.040	2.144	2.453	2.744	3.022	3.375	3.633
32	0.255	0.682	1.309	1.694	1.808	2.037	2.141	2.449	2.738	3.015	3.365	3.622
33	0.255	0.682	1.308	1.692	1.806	2.035	2.138	2.445	2.733	3.008	3.356	3.611
34	0.255	0.682	1.307	1.691	1.805	2.032	2.136	2.441	2.728	3.002	3.348	3.601
35	0.255	0.682	1.306	1.690	1.803	2.030	2.133	2.438	2.724	2.996	3.340	3.591
36	0.255	0.681	1.306	1.688	1.802	2.028	2.131	2.434	2.719	2.990	3.333	3.582
37	0.255	0.681	1.305	1.687	1.800	2.026	2.129	2.431	2.715	2.985	3.326	3.574
38	0.255	0.681	1.304	1.686	1.799	2.024	2.127	2.429	2.712	2.980	3.319	3.566
39	0.255	0.681	1.304	1.685	1.798	2.023	2.125	2.426	2.708	2.976	3.313	3.558
40	0.255	0.681	1.303	1.684	1.796	2.021	2.123	2.423	2.704	2.971	3.307	3.551
60	0.254	0.679	1.296	1.671	1.781	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.254	0.678	1.292	1.664	1.773	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.254	0.677	1.290	1.660	1.769	1.984	2.081	2.364	2.626	2.871	3.174	3.390
120	0.254	0.677	1.289	1.658	1.766	1.980	2.076	2.358	2.617	2.860	3.160	3.373
140	0.254	0.676	1.288	1.656	1.763	1.977	2.073	2.353	2.611	2.852	3.149	3.361
160	0.254	0.676	1.287	1.654	1.762	1.975	2.071	2.350	2.607	2.847	3.142	3.352
180	0.254	0.676	1.286	1.653	1.761	1.973	2.069	2.347	2.603	2.842	3.136	3.345
200	0.254	0.676	1.286	1.653	1.760	1.972	2.067	2.345	2.601	2.838	3.131	3.340
250	0.254	0.675	1.285	1.651	1.758	1.969	2.065	2.341	2.596	2.832	3.123	3.330
inf	0.253	0.674	1.282	1.645	1.751	1.960	2.054	2.326	2.576	2.807	3.090	3.290

Student's t - see page 114 of Holman

You can apply this to analyses done in Lab #2